Vector Kalman Filter

Lets first extend the state model to the vector case while leaving the observation scalar.

$$\underline{s}[n] = A\,\underline{s}[n-1] + B\underline{u}[n] \quad , n \geq 0$$

$\underline{s}$ is a $p \times 1$ state vector

$\underline{u}$ is a $r \times 1$ driving noise vector

A is a $p \times p$ matrix, B is a $p \times r$ matrix (known

$s[-1] \sim N(\mu_s, C_s)$, $u[n] \sim N(0, Q)$,

$s[-1]$ indep of u

ler
ervation $X[n] = \underline{h}^T[n]\,\underline{s}[n] + w[n]$

$\underline{h}[n]$ $p \times 1$ vector (known)

$w[n]$ $N(0, \sigma_n^2)$ WGN indep. of $u[n]$ and $s[-1]$

Example: Stock market

State vector : $\underline{s}[n] = \begin{bmatrix} p[n] \\ r[n] \end{bmatrix}$ $\begin{matrix} \rightarrow \text{true value} \\ \rightarrow \text{daily change rate} \end{matrix}$

$$\underline{s}[n] = \underbrace{\begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}}_{A} \underline{s}[n-1] + \begin{bmatrix} 0 \\ u[n] \end{bmatrix}$$

driving noise is only in daily change rate (market sentiment)

$\Delta = 1$ if n corresponds to day #

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_u^2 \end{bmatrix}$

$$\underline{s}[-1] \sim N\left(\underbrace{\begin{pmatrix} 10000 \\ 0 \end{pmatrix}}_{\mu_s} , \underbrace{\begin{bmatrix} \sigma_p^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}}_{C_s} \right)$$

$u[n]$ is 0-mean with variance σ_u^2

Important, Probably too simplistic assumption

for instance, $\sigma_p^2 = 200^2$

$$x[n] = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{h^T} \underbrace{\begin{bmatrix} p[n] \\ r[n] \end{bmatrix}}_{\underline{s}[n]} + w[n]$$

notice indep. of n.
for this example

WGN
with $\sigma_n^2 = 200$

$$\hat{s}[-1|-1] = \mu_s = \begin{bmatrix} 10000 \\ 0 \end{bmatrix}$$

$$M[-1|-1] = C_s = \begin{bmatrix} 200^2 & 0 \\ 0 & 50^2 \end{bmatrix}$$

Prediction: $\hat{s}[n|n-1] = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \hat{s}[n-1|n-1]$

Min. Prediction MSE: $M[n|n-1] = A\, M[n-1|n-1] A^T + BQB^T$

$$M[n|n-1] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} M[n-1|n-1] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_u \end{bmatrix}$$

Kalman Gain: $K[n] = \dfrac{M[n|n-1] \begin{bmatrix} 1 \\ 0 \end{bmatrix} h}{\sigma_n^2 + \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{h^T} M[n|n-1] \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{h}}$

notice scalar

Correction: $\hat{s}[n|n] = \hat{s}[n|n-1] + K[n]\left(x[n] - h^T \hat{s}[n|n-1]\right)$

Min MSE: $M[n|n] = \left(I - K[n] h^T\right) M[n|n-1]$

Now we will extend the observations to be vector also

$$\underline{x}[n] = \underbrace{H[n]}_{\substack{M \times p \\ \text{matrix}}} \underline{s}[n] + \underbrace{\underline{w}[n]}_{\substack{\text{WGN} \\ \mathcal{N}(0,\, C[n])}}$$

M = dim of observation vector
p = " of state vector

<u>Example</u>: Vehicle tracking (airplane)

State vector $\underline{s}[n] = \begin{bmatrix} p_x[n] \\ p_y[n] \\ v_x[n] \\ v_y[n] \end{bmatrix}$

$$\underline{s}[n] = \overbrace{\begin{bmatrix} 1 & 0 & \Delta & 0 \\ 0 & 1 & 0 & \Delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{A} \underline{s}[n-1] + \begin{bmatrix} 0 \\ 0 \\ u_x[n] \\ u_y[n] \end{bmatrix}$$

Driving noise only in velocity change

Constat velocity model perturbed only by small corrections & noise (wind etc.)

Take $\Delta = \underline{1}$ (unit time intervals)

$$\underline{x}[n] = \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}^{H} \underline{s}[n] + \underbrace{\begin{bmatrix} w_x[n] \\ w_y[n] \end{bmatrix}}_{\text{observation noise}}$$

$M=2$ (observing only position via radar)

$$B = I, \quad Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{u_x}^2 & 0 \\ 0 & 0 & 0 & \sigma_{u_y}^2 \end{bmatrix}$$

Make these equal for this problem

WGN with $C[n] = \begin{bmatrix} \sigma_n^2 & 0 \\ 0 & \sigma_n^2 \end{bmatrix}$

$$s[-1] \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, C_s \right)$$

μ_s is at origin & zero velocity

$$C_s = \begin{pmatrix} 10^2 & 0 & 0 & 0 \\ 0 & 10^2 & 0 & 0 \\ 0 & 0 & 10^2 & 0 \\ 0 & 0 & 0 & 10^2 \end{pmatrix}$$

Prediction: $\hat{s}[n|n-1] = A\,\hat{s}[n-1|n-1]$

Min. Prediction MSE: $M[n|n-1] = A M[n-1|n-1] A^T + B Q B^T$

Kalman gain: $K[n] = M[n|n-1] H^T \left(C[n] + H M[n|n-1] H^T \right)^{-1}$

Correction: $\hat{s}[n|n] = \hat{s}[n|n-1] + K[n]\left(x[n] - H\,\hat{s}[n|n-1] \right)$

Min MSE: $M[n|n] = \left(I - K[n] H \right) M[n|n-1]$

Extended Kalman Filter

non-lineer state ~~or~~ and/or observation equations

$$s[n] = a(s[n-1]) + B u[n]$$

$$x[n] = h(s[n]) + w[n]$$

Linearize $a(s[n-1])$ about $\hat{s}[n-1|n-1]$

Similarly linearize $h(s[n])$

Also, particle filtering is a method to deal with this situation.