1) a) $\underline{s} = \begin{bmatrix} 0 \\ r \\ \vdots \\ r^{N-1} \end{bmatrix} \qquad C = \sigma^2 I$

$$\hat{A}_{BLUE} = \frac{\underline{s}^T C^{-1} \underline{x}}{\underline{s}^T C^{-1} \underline{s}} = \frac{\sum r^n x[n]}{\sum r^{2n}}$$

b) $$\mathrm{var}(\hat{A}_{BLUE}) = \frac{1}{\underline{s}^T C^{-1} \underline{s}} = \frac{1}{\sum_{n=0}^{N-1} r^{2n}}$$

$$\lim_{N \to \infty} \frac{1}{\sum_{n=0}^{N-1} r^{2n}} = 0 \quad \text{if} \quad |r| \geq 1$$

c) $w[n]$ must be Gaussian

2) a) $$p(\underline{x}; \lambda) = \lambda^N \exp\left[-\lambda \sum x[n] \right]$$

$$\ln p = N \ln \lambda - \lambda \sum x[n]$$

$$\frac{\partial \ln p}{\partial \lambda} = \frac{N}{\lambda} - \sum x[n] = 0$$

$$\hat{\lambda} = \frac{N}{\sum x[n]}$$

b) $\hat{\lambda} \overset{a}{\sim} N(\lambda, I^{-1}(\lambda)) \qquad I(\lambda) = -E\left[\frac{\partial^2 \ln p}{\partial \lambda^2} \right]$

$$= -E\left[-N/\lambda^2 \right]$$

$\hat{\lambda} \overset{a}{\sim} N\left(\lambda, \frac{\lambda^2}{N}\right) \qquad \Longleftarrow \qquad = N/\lambda^2$

3 a) $J(\theta) = \sum_{n=0}^{N-1} \left(x[n] - \theta h[n] \right)^2$

$$= \sum_{n=0}^{N/2-1} (x[n]+\theta)^2 + \sum_{n=N/2}^{N-1} (x[n]-\theta)^2$$

$$\frac{\partial J}{\partial \theta} = 2\sum_{n=0}^{N/2-1} (x[n]+\theta) - 2\sum_{n=N/2}^{N-1} (x[n]-\theta) = 0$$

$$\hat{\theta} = \frac{-\sum_{n=0}^{N/2-1} x[n] + \sum_{n=N/2}^{N-1} x[n]}{N}$$

b) $\underline{x} = \underline{H}\,\underline{\theta} + \underline{w}$ $\quad \underline{\theta} = \begin{pmatrix} \theta \\ \beta \end{pmatrix}$ $\quad H = \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ \vdots & 1 \\ -1 & \vdots \\ 1 & \vdots \\ \vdots & \\ 1 & 1 \end{bmatrix}$

$$\underline{\hat{\theta}} = (H^T H)^{-1} H^T x \qquad H^T H = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}$$

$$\underline{\hat{\theta}} = \begin{pmatrix} 1/N & 0 \\ 0 & 1/N \end{pmatrix} H^T x$$

$$\underline{\hat{\theta}} = \begin{pmatrix} -\sum_{0}^{N/2-1} x[n] + \sum_{N/2}^{N-1} x[n] \\ \sum_{0}^{N-1} x[n] \end{pmatrix} / N$$

4) a) $\mu_1 = \alpha A + (1-\alpha) A = A$

$$\hat{A} = \hat{\mu}_1 = \frac{1}{N} \sum x[n]$$

$$\mu_2 = \text{var} + \mu_1^2 = \alpha \cdot 1 + (1-\alpha) \cdot 2 + A^2$$

$$\mu_2 = 2 - \alpha + A^2$$

$$\hat{\mu}_2 = 2 - \alpha + \hat{A}^2$$

$$\hat{\alpha} = \hat{A}^2 + 2 - \hat{\mu}_2$$

$$= \left(\frac{1}{N} \sum x[n]\right)^2 + 2 - \frac{1}{N} \sum x[n]^2$$

b) when $y[n]$ known

$$p(x[n]; A) = \begin{cases} N(A,1) & y[n] = 0 \\ N(A,2) & y[n] = 1 \end{cases}$$

Let Ω_0 be the set for which $y[n] = 0$
Let Ω_1 " " " " $y[n] = 1$

Then

$$p(\underline{x}; A) = \prod_{n \in \Omega_0} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x[n]-A)^2}{2}\right] \prod_{n \in \Omega_1} \frac{1}{\sqrt{4\pi}} \exp\left[\frac{}{4}\right]$$

$$\ln p = \sum_{n \in \Omega_0} (x[n] - A) + \sum_{n \in \Omega_1} \frac{(x[n] - A)}{2} + \text{terms indep of } A ..$$

$$\frac{\partial \ln p}{\partial A} = \sum_{n \in \Omega_0} x[n] - \underbrace{|\Omega_0|}_{\text{num elements in } \Omega_0} A + \frac{1}{2} \sum_{n \in \Omega_1} x[n] - \frac{|\Omega_1| A}{2} = 0$$

$$\sum_{n \in \Omega_0} x[n] + \frac{1}{2} \sum_{n \in \Omega_1} x[n] = A\left(|\Omega_0| + \frac{|\Omega_1|}{2} \right)$$

$$\hat{A} = \frac{\displaystyle\sum_{n \in \Omega_0} x[n] + \frac{1}{2} \sum_{n \in \Omega_1} x[n]}{|\Omega_0| + \frac{|\Omega_1|}{2}}$$

This estimator is giving 1/2 the weight to samples which have variance 2.