METHOD OF MOMENTS

Assume k'th moment $\mu_k = E[x^k]$ depends on unknown parameter θ according to $\mu_k = h(\theta)$.

Then we solve for : $\theta = h^{-1}(\mu_k)$

Then replace the theoretical moment by its natural estimator $\hat{\mu}_k = \frac{1}{N}\sum_{n=0}^{N-1} x^k[n]$

Finally $\hat{\theta} = h^{-1}\left(\frac{1}{N}\sum_{n=0}^{N-1} x^k[n]\right)$

Example 1 : N IID observations from Laplacian

PDF $p(x;\sigma) = \frac{1}{\sqrt{2}\sigma}\exp\left[-\frac{\sqrt{2}|x|}{\sigma}\right]$

Which moment can we use to estimate σ?

$\mu_0 = \frac{1}{N}\sum x^0[n] = 1$ useless $E[x^0]$

$\mu_1 = E[x] = 0$ useless

$\mu_2 = E[x^2] = \int \frac{x^2}{\sqrt{2}\sigma} e^{-\frac{\sqrt{2}|x|}{\sigma}} dx$

$= \frac{2}{\sqrt{2}\sigma}\int_0^\infty x^2 e^{-\sqrt{2}x/\sigma} dx = \sigma^2$

so $\sigma = \sqrt{E[x^2]}$

$\hat{\sigma} = \sqrt{\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]}$

Example 2 : N IID observations from exponential

PDF $p(x[n];\lambda) = \begin{cases} \lambda\exp[-\lambda x[n]], & x[n] \geq 0 \\ 0, & \text{otherwise} \end{cases}$

$$\mu_1 = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}\int_0^\infty y e^{-y} dy$$

let $y = x\lambda$

$dy = \lambda dx$

$dx = \frac{1}{\lambda} dy$

$$= 1/\lambda$$

Therefore $\hat{\lambda} = \dfrac{1}{\underbrace{\frac{1}{N}\sum x[n]}_{\hat{\mu}_1}}$

Example 3 : N IID samples from bivariate Gaussian PDF = $\underline{x}_0, \underline{x}_1, \ldots \underline{x}_{N-1}$ where each $\underline{x}_k$ is a random vector with Normal distribution $N(0, C)$ where $C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$. How do we estimate ρ using moments.

$$\rho = \text{cov}(u, v) \quad \text{where} \quad \underline{x}_k = \begin{bmatrix} u[k] \\ v[k] \end{bmatrix}$$

$$\hat{\rho} = \frac{1}{N}\sum_{k=0}^{N-1} u[k]v[k]$$

Example 4 : Gaussian mixture

$$x[n] \sim \varepsilon N(0, \sigma_1^2) + (1-\varepsilon) N(0, \sigma_2^2)$$

σ_1, σ_2 known $\quad \varepsilon$ unknown

$$E[x^2[n]] = \varepsilon[0^2 + \sigma_1^2] + (1-\varepsilon)[0 + \sigma_2^2]$$

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] = \varepsilon \sigma_1^2 + (1-\varepsilon)\sigma_2^2 = \varepsilon(\sigma_1^2 - \sigma_2^2) + \sigma_2^2$$

$$\varepsilon = \frac{\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] - \sigma_2^2}{\sigma_1^2 - \sigma_2^2}$$

Example 4 continued

Now σ_1, σ_2 and ε unknown (Vector parameter)

$$\mu_2 = E\left[x^2[n]\right] = \varepsilon\sigma_1^2 + (1-\varepsilon)\sigma_2^2$$

$$\mu_4 = E\left[x^4[n]\right] = 3\varepsilon\sigma_1^4 + 3(1-\varepsilon)\sigma_2^4$$

$$\mu_6 = E\left[x^6[n]\right] = 15\varepsilon\sigma_1^6 + 15(1-\varepsilon)\sigma_2^6$$

3 equations, 3 unknowns.

estimate μ_2, μ_4, μ_6 as

$$\hat{\mu}_k = \frac{1}{N}\sum_{n=0}^{N-1} x^k[n]$$

let $u = \sigma_1^2 + \sigma_2^2$

$$u = \frac{\mu_6 - 5\mu_4\mu_2}{5\mu_4 - 15\mu_2^2}$$

let $v = \sigma_1^2\sigma_2^2$

$$v = \mu_2 u - \mu_4/3$$

Then $\sigma_1^2 = \dfrac{u + \sqrt{u^2 - 4v}}{2}$

$$\sigma_2^2 = v/\sigma_1^2$$

and $\varepsilon = \dfrac{\sigma_2^2 - \mu_2}{\sigma_2^2 - \sigma_1^2}$

Example 5 $\quad x[n] \sim U[0, \beta]$

$$\mathrm{var}(x[n]) = \beta^2/12$$

$$\hat{\beta} = \sqrt{\frac{\frac{1}{N}\sum_{n=0}^{N-1}(x^2 - \bar{x})^2}{12}}$$ where $\bar{x} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$

Example 6 $\quad x[n] \sim U[a,b]$

$$\mu_1 = E[x[n]] = \frac{a+b}{2}$$

$$\mu_2 = E[x^2[n]] = \underbrace{\left(\frac{a+b}{2}\right)^2}_{\mu_1^2} + \underbrace{var(x[n])}_{\frac{(a-b)^2}{12}}$$

so $\hat{\mu}_1 = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$, $\hat{\mu}_2 = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$

then $\frac{(a-b)^2}{12} = \hat{\mu}_2 - (\hat{\mu}_1)^2$

$$b-a = \sqrt{12(\hat{\mu}_2 - (\hat{\mu}_1)^2)} \quad \text{since } b>a$$

then
$$\hat{a} = \hat{\mu}_1 - \frac{1}{2}\sqrt{12(\hat{\mu}_2 - \hat{\mu}_1^2)}$$
$$\hat{b} = \hat{\mu}_1 + \frac{1}{2}\sqrt{12(\hat{\mu}_2 - \hat{\mu}_1^2)}$$

Example 7 $\quad x[n] \sim \frac{1}{2}U[0,\beta] + \frac{1}{2}N(0,\sigma^2)$

mixture of uniform and Gaussian distributions

$$\mu_1 = E[x] = \frac{1}{2}E[U[0,\beta]] + \underbrace{\frac{1}{2}E[N(0,\sigma^2)]}_{0}$$
$$= \frac{1}{4}\beta$$

$$\mu_2 = E[x^2] = \frac{1}{2}\frac{\beta^2}{12} + \frac{1}{2}\sigma^2$$

so $\hat{\mu}_1 = \frac{1}{N}\sum x[n] \Rightarrow \hat{\beta} = 4\hat{\mu}_1$

$$\hat{\sigma}^2 = 2\left[\hat{\mu}_2 - \frac{(4\hat{\mu}_1)^2}{24}\right]$$

where $\hat{\mu}_2 = \frac{1}{N}\sum x^2[n]$