TUTOR: THE PHASOR TRANSFORM

All voltages and currents in linear circuits with sinusoidal sources are described by constant-coefficient linear differential equations of the form

(1) $$a_n \frac{d^n f}{dt^n} + a_{n-1} \frac{d^{n-1} f}{dt^{n-1}} + \ldots + a_0 f = C \ \cos(\omega t + \phi)$$

where f is a function of time, the a_n are constants, C is a constant, ω is the radian frequency of the sinusoidal source, and ϕ is the phase of the sinusoidal source. In (1), f represents any voltage or current in the circuit.

A particular solution to (1) can be found by an elegant procedure called the **phasor transform** method. This supplementary material outlines the mathematical basis of the method. The phasor transform is defined by

(2) $$f(t) = \text{Re}\left[\mathbf{F}(\omega) e^{j\omega t} \right]$$

where $\mathbf{F}(\omega)$ is a function of ω called the **phasor transform** of f(t), and Re means the real part of the quantity in the brackets. $\mathbf{F}(\omega)$ is complex; it has a real and an imaginary part.

Two key mathematical relationships are used in finding a particular solution to (1). The first is

(3) $$\text{Re}[W] = \frac{W + W*}{2}$$

where W is any complex number and W* is the complex conjugate of W. Using (3) with (2) gives

(4) $$f = \frac{\mathbf{F} e^{j\omega t} + \mathbf{F}* e^{-j\omega t}}{2}$$

where f has been written for f(t) and **F** for $\mathbf{F}(\omega)$ for brevity. Note that **F** is **not** a function of time. The second relationship is

(5) $\qquad \cos(\omega t + \phi) = \dfrac{e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)}}{2}$

which is called Euler's formula.

Substituting (5) and (4) into (1), taking the derivatives with respect to time, and collecting terms gives

(6) $\quad \left[a_n (j\omega)^n \mathbf{F} + a_{n-1}(j\omega)^{n-1}\mathbf{F} + \ldots + a_0 \mathbf{F} - Ce^{j\phi} \right] e^{j\omega t} +$

$\qquad \left[a_n(-j\omega)^n \mathbf{F}* + a_{n-1}(-j\omega)^{n-1}\mathbf{F}* + \ldots + a_0 \mathbf{F}* - Ce^{-j\phi} \right] e^{-j\omega t} = 0.$

Now because $e^{j\omega t}$ and $e^{-j\omega t}$ are **linearly independent** functions (see, for example, C. R. Wylie, *Advanced Engineering Mathematics*, 3rd ed., New York: McGraw-Hill, 1966, p. 444), (6) can be true for all time only if

(7) $\quad \left[a_n (j\omega)^n \mathbf{F} + a_{n-1}(j\omega)^{n-1}\mathbf{F} + \ldots + a_0 \mathbf{F} - Ce^{j\phi} \right] = 0$

and

(8) $\quad \left[a_n(-j\omega)^n \mathbf{F}* + a_{n-1}(-j\omega)^{n-1}\mathbf{F}* + \ldots + a_0 \mathbf{F}* - Ce^{-j\phi} \right] = 0.$

Equations (7) and (8) are identical because one is the complex conjugate of the other, so only one is needed. An expression for **F** from (7) is

(9) $\qquad \mathbf{F} = \dfrac{Ce^{j\phi}}{a_n (j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_0}.$

A particular solution to (1) can now be obtained from (9) and (2):

(10) $\qquad f = \mathrm{Re}\left[\mathbf{F}e^{j\omega t} \right] = \mathrm{Re}\left[\dfrac{Ce^{j\phi}e^{j\omega t}}{a_n (j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_0} \right].$

Symbolically, the notation for a phasor transformation is

(11) $\mathbf{P}[f(t)] = \mathbf{F}(\omega)$

where the bold $\mathbf{P}$ means "phasor transform of". Thus, $\mathbf{F}$ is the phasor transform of f. Taking the derivative of both sides of (2) gives

$$\frac{df}{dt} = \mathrm{Re}\left[j\omega \mathbf{F}(\omega)e^{j\omega t} \right]$$

which corresponds to

$$\mathbf{P}\left[\frac{df}{dt}\right] = j\omega \mathbf{F}.$$

Similarly,

(12) $\mathbf{P}\dfrac{d^n f}{dt^n} = \left(j\omega\right)^n \mathbf{F}$

and

(13) $\mathbf{P}\left[\cos(\omega t + \phi)\right] = e^{j\phi}$

because

(14) $\cos(\omega t + \phi) = \mathrm{Re}\left[e^{j\phi}e^{j\omega t}\right].$

From the basic relation in (2) it can also be shown that

(15) $\mathbf{P}\left[f_1 + f_2\right] = F_1 + F_2$

and

(16) $\mathbf{P}[af] = a\mathbf{F}$

where

$\mathbf{P}[f_1] = \mathbf{F}_1$

and

$\mathbf{P}[f_2] = \mathbf{F}_2$

and "a" is a constant. The relation in (15) means that the phasor transform of a sum of functions can be found by taking the transform of each function and adding the transforms.

Equations (11), (12), (13), (15), and (16) describe phasor transforms. An inverse phasor transform relation is written as

(17) $f(t) = \mathbf{P}^{-1}[\mathbf{F}(\omega)]$.

Equations (11) and (17) are called a **transform pair**. Equation (11) states how to get $\mathbf{F}$ when f is known; (17) how to get f when $\mathbf{F}$ is known. Equation (2) is the inverse transform relation. The transform relation is derived as follows. f(t) will always be a sinusoid, because it is a particular solution to (1). Thus f(t) can be written as

(18) $f(t) = f_m \cos(\omega t + \alpha)$.

Substituting (18) into (2), using Euler's formula and (3) gives

$$\frac{f_m e^{j\omega t} e^{j\alpha} + f_m e^{-j\omega t} e^{-j\alpha}}{2} = \frac{\mathbf{F} e^{j\omega t} + \mathbf{F}^* e^{-j\omega t}}{2}.$$

Collecting terms and using the linear independence of $e^{j\omega t}$ and $e^{-j\omega t}$, as before, gives

(19) $\mathbf{F} = f_m e^{j\alpha}$

so the phasor transform of $f_m \cos(\omega t + \alpha)$ is $f_m e^{\alpha t}$. The transform pairs are thus

(20) $\mathbf{F} = \mathbf{P}[f] = f_m e^{j\alpha}$

and

(21) $f = \mathbf{P}^{-1}[\mathbf{F}] = \text{Re}\left[\mathbf{F}e^{j\omega t}\right].$

With the phasor transform relations given in (12), (15, (16), (20), and (21), a particular solution to (1) can be found without going through the detailed derivation using (3) and linear independence. The phasor transform of (1) is taken term-by-term using (12), (13), (15), and (16) to get (7), which is then solved for $\mathbf{F}$. Having found $\mathbf{F}$, f is found by taking the inverse transform according to (21).

EX: Let's find a particular solution to

(22) $\dfrac{d^3 f}{dt^3} + 3\dfrac{d^2 f}{dt^2} + 50\dfrac{df}{dt} - 60f = 500\cos(10t + \pi/3).$

Taking the phasor transform of this equation gives

$$(j10)^3 \mathbf{F} + 3(j10)^2 \mathbf{F} + 50(j10)\mathbf{F} - 60\mathbf{F} = 500 e^{j\pi/3}.$$

Solving for $\mathbf{F}$,

$$\mathbf{F} = \frac{500 e^{j\pi/3}}{-j100 - 300 + j500 - 60}.$$

Converting $\mathbf{F}$ to polar form gives

$$\mathbf{F} = 0.812 e^{-j174.25°}$$

and finding the inverse transform gives

$$f = \mathrm{Re}\left[\mathbf{F}e^{j\omega t}\right] = \mathrm{Re}\left[0.812 e^{-j174.25°} e^{j10t}\right] = 0.812\cos(10t - 174.25°).$$

COMMENT: The phasor transform method is powerful because it transforms a differential equation (1) into an algebraic equation (7), which can be solved for the phasor **F**, and then f can be found by taking the inverse transform.

Phasor voltages and currents satisfy Kirchhoff's laws, because of (15). Consequently, circuits can be transformed into the frequency domain, eliminating the need to write differential equations in the time domain and solve them by phasor transforms. The procedure for analyzing and designing circuits by transforming them into the frequency domain is summarized in the figure below. Note that impedance is defined as the ratio of a phasor voltage to a phasor current. Impedance is not defined in the time domain.

TIME DOMAIN   **FREQUENCY DOMAIN**

phasor transform

Frequency-domain representation of
the circuit (transformed circuit)

Apply circuit laws to obtain
algebraic equations:

$$\mathbf{V}_g = R\,\mathbf{I} + j\,\omega L\,\mathbf{I}$$

Solve for the desired quantities:

$$\mathbf{I} = \frac{\mathbf{V}_g}{R + j\,\omega L}$$

$$i = \frac{v_m \cos\,(\omega t + \phi - \psi)}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\psi = \tan^{-1}\,(\omega L/R)$$

Time-domain expression
for the desired quantities

Inverse phasor transform