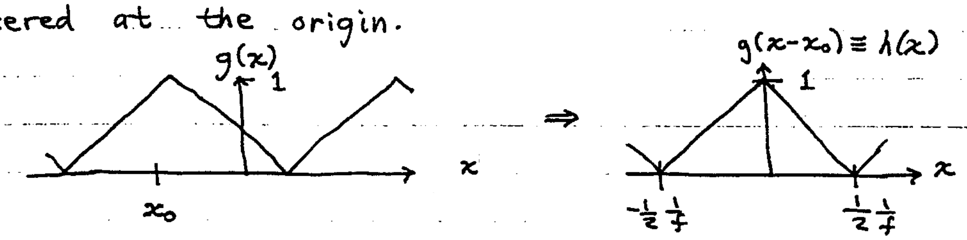
Fourier Series – Coefficients – Calculation

comment: We apply the following tools to find Fourier coefficients

tool: We can shift a periodic function so that it is centered at the origin.

ex: 

$g(x) = \Lambda(x)$ $\qquad\qquad$ $g(x-x_0) = \Lambda(x-x_0)$

Now we can find the Fourier series for $g(x-x_0) = h(x)$ and shift back to the original coordinates later on.

tool: If a function is even, then the coefficients of sin terms in Fourier series are zero: $h(x) = h(-x) \Rightarrow h(x) = \sum_{n=0}^{\infty} a_n \sqrt{2f}\cos 2\pi n f x$

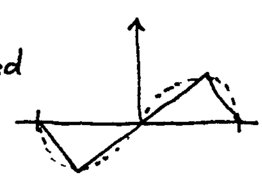
$b_n = 0$ for all n.

ex: For the example above this tool applies to $h(x)$.

tool: If a function is odd, then the coefficients of cos terms in the Fourier series are zero:

$$h(x) = -h(x) \Rightarrow h(x) = \sum_{n=0}^{\infty} b_n \sqrt{2f}\sin 2\pi n f x$$

ex: odd function with sin superimposed



Fourier Series – Coefficients – Calculation (cont.)

tool: We find Fourier coeff for a func h from inner products.

$$h = \sum_{n=0}^{\infty} a_n \sqrt{2f} \cos 2\pi n f x + b_n \sqrt{2f} \sin 2\pi n f x$$

$$a_n = (h, \sqrt{2f} \cos 2\pi n f x) = \int_{-\frac{1}{2}\frac{1}{f}}^{\frac{1}{2}\frac{1}{f}} h \sqrt{2f} \cos 2\pi n f x \, dx$$

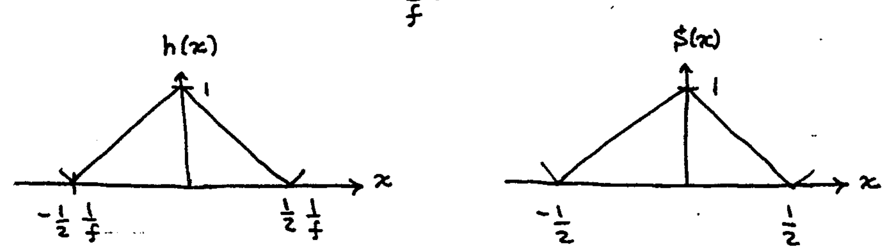
$$b_n = (h, \sqrt{2f} \sin 2\pi n f x) = \int_{-\frac{1}{2}\frac{1}{f}}^{\frac{1}{2}\frac{1}{f}} h \sqrt{2f} \sin 2\pi n f x \, dx$$

where h has period $1/f$.

tool: We can scale the x axis to make the period of a function equal to 1. Then we can compute the Fourier series of this function and scale the result to get Fourier coefficients for the original function.

$$h(x) \Rightarrow h\left(\frac{x}{f}\right) = s(x)$$

ex:



tool: $h = \sum_{n=0}^{\infty} a_n \sqrt{2f} \cos 2\pi n f x + b_n \sqrt{2f} \sin 2\pi n f x$ $\qquad$ $h\left(\frac{x}{f}\right) = s = \sum_{n=0}^{\infty} \alpha_n \sqrt{2} \cos 2\pi n x + \beta_n \sqrt{2} \sin 2\pi n x$

Because magnitudes must agree, we must have the following relationship for h and s.

$$a_n \sqrt{2f} = \alpha_n \sqrt{2},$$

$$a_n = \frac{\alpha_n}{\sqrt{f}}.$$

Now shift the origin back to its original position.

— Calculation

Fourier Series — Coefficients ^ (cont.)

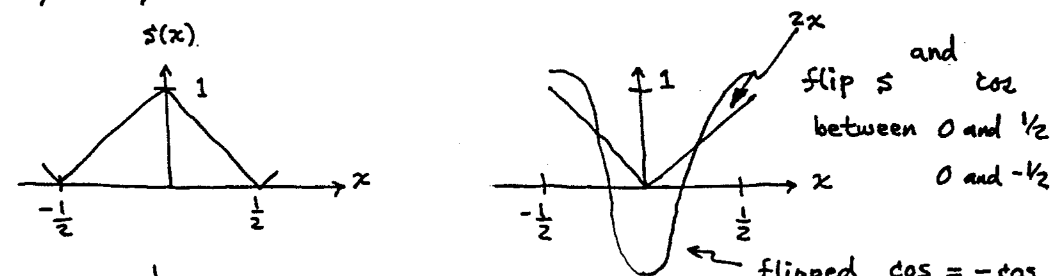
tool: We can often manipulate integrals in the following ways to simplify them:

1) Flip the integrated function along the x axis
2) Shift the " " " " "
3) Adjust the integration limits so as to keep ^constant the area under the curve being integrated.

This procedure amounts to the observation that an integral is the area under a function between two points along the x-axis.

So long as we keep the area under the curve the same, we can manipulate the integral in any way we wish.

ex:



$$a_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} s\,\sqrt{2}\,\cos 2\pi n x \; dx$$

$$= 2\int_{0}^{\frac{1}{2}} 2x\,\sqrt{2}\,(-\cos 2\pi n x)\; dx$$

$$= -4\sqrt{2}\int_0^{1/2} x\cos 2\pi n x \; dx$$

tool: We can compute the area under a symmetric curve by doubling the integral for one side, (as in the above example).

tool: We can ^change variables to simplify integrals. We use two steps

1) Multiply by factors to make all variables look the same.
2) We change the limits of integration for the new variable.

Fourier Series — Coefficients – Calculation (cont.)

ex: (cont.)

$$d_n = -4\sqrt{2}\int_0^{1/2} x \cos 2\pi n x \, dx$$

$$= -4\sqrt{2}\int_0^{1/2} \frac{2\pi n x}{2\pi n} \cos 2\pi n x \, \frac{d(2\pi n x)}{2\pi n}$$

$$= \frac{-4\sqrt{2}}{(2\pi n)^2}\int_0^{1/2} 2\pi n x \cos 2\pi n x \; d\, 2\pi n x$$

Define $y = 2\pi n x$ $\quad \frac{dy}{dx} = 2\pi n \Rightarrow dy = 2\pi n \, dx = d\, 2\pi n x$

Limits: $x = 0 \Rightarrow y = 2\pi n \cdot 0 = 0$

$x = \frac{1}{2} \Rightarrow y = 2\pi n \cdot \frac{1}{2} = \pi n$

$$d_n = \frac{-4\sqrt{2}}{(2\pi n)^2}\int_0^{\pi n} y \cos y \, dy$$

From $\int$ table $\quad \int y \cos y = \cos y + y \sin y$

$$d_n = \frac{-4\sqrt{2}}{(2\pi n)^2}\left(\cos y + y \sin y\right)\Big|_0^{\pi n}$$

$$= \begin{cases} \frac{-4\sqrt{2}}{(2\pi n)^2}\left[(-1 + \pi n \cdot 0) - (1 + 0)\right] & n \cdot \text{odd} \\ \frac{-4\sqrt{2}}{(2\pi n)^2}\left[(1 + \pi n \cdot 0) - (1 + 0)\right] = 0 & n \text{ even} \end{cases}$$

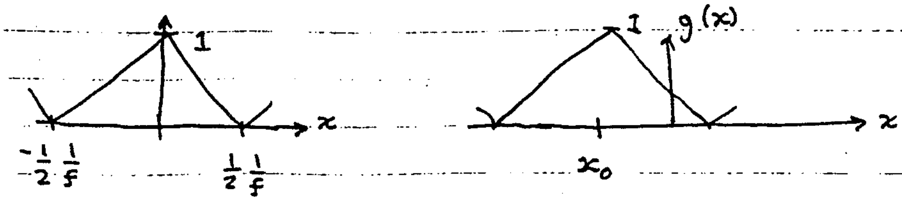
$$d_{n\,odd} = \frac{8\sqrt{2}}{(2\pi n)^2} \qquad d_{n\,even} = 0$$

Now scale the x-axis back to the original

$$a_{n\,odd} = \frac{\alpha_{n\,odd}}{\sqrt{f}} \qquad a_{n\,even} = 0$$

Fourier Series – Coefficients – Calculation (cont.)

Now shift origin back to original position



tool: We want sinusoid centered at 0 to shift to x_0. Thus, we want the argument of the sinusoid to be zero at x_0:

before shift		after shift
$a_n\sqrt{2f}\cos 2\pi nfx$	$\longrightarrow$	$a_n\sqrt{2f}\cos 2\pi nf(x-x_0)$
$b_n\sqrt{2f}\sin 2\pi nfx$	$\longrightarrow$	$b_n\sqrt{2f}\sin 2\pi nf(x-x_0)$

tool: Use trigonometric identities to rewrite sinusoids after they are shifted back to original origin.

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

ex: (cont.) before identity → after identity

$$a_n\sqrt{2f}\cos 2\pi nf(x-x_0) \longrightarrow a_n\sqrt{2f}\cos 2\pi nfx\cos 2\pi nfx_0 + a_n\sqrt{2f}\sin 2\pi nfx\sin 2\pi nfx_0$$

$$b_n\sqrt{2f}\sin 2\pi nf(x-x_0) \longrightarrow b_n\sqrt{2f}\sin 2\pi nfx\cos 2\pi nfx_0 - b_n\sqrt{2f}\cos 2\pi nfx\sin 2\pi nfx_0$$

Using previously derived values for a_n and b_n and recombining terms gives the final answer: (we had only cos terms)

$$g(x) = \sum_{n=0}^{\infty} A_n\sqrt{2f}\cos 2\pi nfx + B_n\sqrt{2f}\sin 2\pi nfx$$

$$A_n = \frac{8\sqrt{2}}{(2\pi n)^2}\frac{1}{\sqrt{f}}\sqrt{2f}\cos 2\pi nfx_0 \quad n \text{ odd}, \; = 0 \quad n \text{ even} \neq 0, \; = \tfrac{1}{2} \quad n=0$$

$$B_n = \frac{8\sqrt{2}}{(2\pi n)^2}\frac{1}{\sqrt{f}}\sqrt{2f}\sin 2\pi nfx_0 \quad n \text{ odd}, \; = 0 \quad n \text{ even}$$

note: A_0 terms is average value of $g(x)$