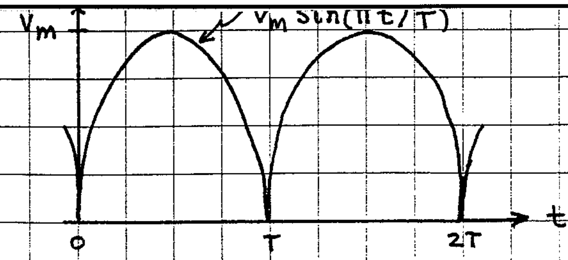


Find Fourier series for $v(t)$.

ans: $$v(t) = \frac{2V_m}{\pi}\left[1 + \sum_{k=1}^{\infty} \frac{2}{1-4k^2} \cos k\omega_o t\right] \qquad \omega_o \equiv 2\pi/T$$

soln: $$v(t) = a_v + \sum_{k=1}^{\infty} a_k \cos k\omega_o t + b_k \sin k\omega_o t \qquad \omega_o \equiv 2\pi/T$$

Note: Text uses n instead of k as the summation index, but then the text gives the coefficient formula for a_k, etc., instead of a_n. This is confusing and unnecessary. I will use k for all subscripts and indices.

$$a_v = \frac{1}{T}\int_0^T v(t)\,dt = \frac{1}{T}\int_0^T V_m \sin\frac{\pi t}{T}\,dt$$

Note: The frequency of the $v(t)$ lobes is $\omega_o/2$. We can exploit some orthogonality results for sinusoids at differing frequencies, but we have only one-half of a cycle and its frequency is lower than the fundamental frequency, ω_o. Thus, we have to proceed with caution.

$$a_v = \frac{V_m}{T}\int_0^T \sin\left(\frac{\pi t}{T}\right) dt$$

$$= \frac{V_m}{\cancel{T}} \left.\frac{-\cos\frac{\pi t}{T}}{\frac{\pi}{\cancel{T}}}\right|_0^T$$

By chain rule,

$$\frac{d}{dt}\cos\frac{\pi t}{T} = -\sin\frac{\pi t}{T}\frac{d}{dt}\frac{\pi t}{T}$$

$$= -\frac{\pi}{T}\sin\frac{\pi}{T}t$$

$$= \frac{-V_m}{\pi}\left(\cos\frac{\pi \cancel{T}}{\cancel{T}} - \cos\frac{\pi\cdot 0}{T}\right)$$

Thus, we divide by π/T to eliminate unwanted π/T factor.

$$= -\frac{V_m}{\pi}(-1-1) = \frac{2V_m}{\pi}$$

$$a_k = \frac{2}{T}\int_0^T v(t)\cos(k\omega_0 t)\,dt \qquad \omega_0 \equiv 2\pi/T$$

$$= \frac{2}{T}\int_0^T V_m \sin\left(\frac{\pi}{T}t\right)\cos\left(k\frac{2\pi}{T}t\right)dt$$

Introduce change of variables to simplify notation:

$\theta \equiv \frac{2\pi t}{T}$ We have $\frac{d\theta}{dt} = \frac{2\pi}{T}$ or $d\theta = \frac{2\pi}{T}dt$

or $dt = \frac{T}{2\pi}d\theta$

$t=0$ (lower limit of $\int$) $\Rightarrow$ $\theta = \frac{2\pi}{T}\cdot 0 = 0$

$t=T$ (upper " " ") $\Rightarrow$ $\theta = \frac{2\pi}{T}T = 2\pi$

Now make direct substitutions for: limits of $\int$, arguments of sin() and cos(), and dt.

$$a_k = \frac{2}{T}\int_0^{2\pi} V_m \sin\left(\frac{\theta}{2}\right)\cos(k\theta)\,\frac{T}{2\pi}d\theta$$

$$= \frac{V_m}{\pi}\int_0^{2\pi}\sin\left(\frac{\theta}{2}\right)\cos(k\theta)\,d\theta$$

Now we may proceed in one of three (or more) ways:

1) We can look up $\int \sin ax \cos bx$ on p. 1009 of text.
2) We can transform $\sin\alpha\cos\beta$ to $\frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$ using identities on p. 1007 of text.
3) We can write $\sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}$, $\cos\beta = \frac{e^{j\beta} + e^{-j\beta}}{2}$,

and integrate the complex exponentials as $\int e^{(a+jb)x}dx = \frac{e^{(a+jb)x}}{a+jb}$. Follows same rule as $\int e^{\alpha x} = \frac{e^{\alpha x}}{\alpha}$ for α real.

Which method we choose is a matter of personal preference. (3) is especially useful if we lack a book of tables.

We'll try method (1) first, (and look up the integral):

$$\int \sin ax \cos bx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

Note: We require $a^2 \neq b^2$ because, otherwise, $a-b=0$ or $a+b=0$ appear in the denominators. If $a = \pm b$ we must use method (2) or (3), as we'll see, below.

We have $a \equiv \frac{1}{2}$ $\quad b \equiv k$ for our application of this integral.

$$\therefore a_k = \frac{V_m}{\pi}\int_0^{2\pi} \sin\overset{a\theta}{\left(\frac{\theta}{2}\right)} \overset{b\theta}{\cos(k\theta)}\, d\theta$$

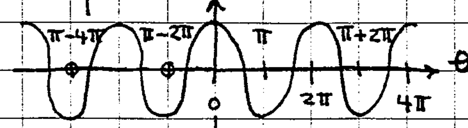
$$= \frac{V_m}{\pi}\left[\frac{-\cos\left(\frac{\theta}{2}-k\theta\right)}{2\left(\frac{1}{2}-k\right)} - \frac{\cos\left(\frac{\theta}{2}+k\theta\right)}{2\left(\frac{1}{2}+k\right)}\right]\Bigg|_0^{2\pi}$$

Note that, since k is an integer, we always, (i.e. for all k), satisfy the $a^2 \neq b^2$ (i.e. $\left(\frac{1}{2}\right)^2 \neq (k)^2$) requirement for the integral.

Move minus signs out front:

$$a_k = -\frac{V_m}{2\pi}\left[\frac{\cos\left(\frac{\theta}{2}-k\theta\right)}{\frac{1}{2}-k} + \frac{\cos\left(\frac{\theta}{2}+k\theta\right)}{\frac{1}{2}+k}\right]\Bigg|_0^{2\pi}$$

Now $\cos\left(\frac{\theta}{2}-k\theta\right)\Big|^{2\pi} = \cos(\pi - k2\pi)$



From the plot of $\cos\theta$ we see that $\cos(\pi - k2\pi) = -1$ for all k. $\therefore \cos\left(\frac{\theta}{2} - k\theta\right)\Big|^{\pi} = -1$ for all k

Similarly, we find that $\cos\left(\frac{\theta}{2} + k\theta\right)\Big|^{\pi} = -1$ for all k.

We also have $\cos\left(\frac{\theta}{2} - k\theta\right)\Big|_0 = \cos 0 = 1$

$$\cos\left(\frac{\theta}{2} + k\theta\right)\Big|_0 = \cos 0 = 1.$$

$$\therefore a_k = \frac{-V_m}{\cancel{2}\pi}\left(\frac{-1}{\frac{1-2k}{\cancel{2}}} - \frac{1}{\frac{1-2k}{\cancel{2}}} \quad \frac{-1}{\frac{1+2k}{\cancel{2}}} - \frac{1}{\frac{1+2k}{\cancel{2}}}\right)$$

$$= \frac{V_m}{\pi}\left(\frac{2}{1-2k} + \frac{2}{1+2k}\right)$$

$$= \frac{2V_m}{\pi}\left(\frac{1}{1-2k} + \frac{1}{1+2k}\right)$$

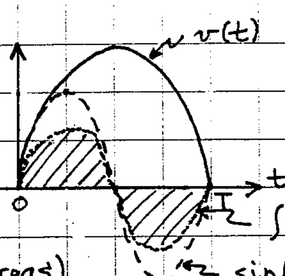
$$= \frac{2V_m}{\pi}\left(\frac{1+2k}{(1-2k)(1+2k)} + \frac{1-2k}{(1+2k)(1-2k)}\right)$$

$$= \frac{2V_m}{\pi}\,\frac{2}{1-4k^2}$$

$$a_k = \frac{4V_m}{\pi(1-4k^2)}$$

Now we consider $b_k = \frac{2}{T}\int_0^T v(t)\sin(k\omega_0 t)\,dt$.

Picture for $k=1$.



Since $v(t)$ is even, (i.e. symmetrical about vertical axis at $t=0$), all $b_k = 0$.

$\int v(t)\sin(\omega_0 t)\,dt = 0$ [equal positive and negative areas]

Similar story, (equal positive and negative areas), so $\int v(t)\sin(k\omega_0 t)\,dt = 0$ for all k.

$\boxed{\therefore b_k = 0 \text{ for all } k}$.