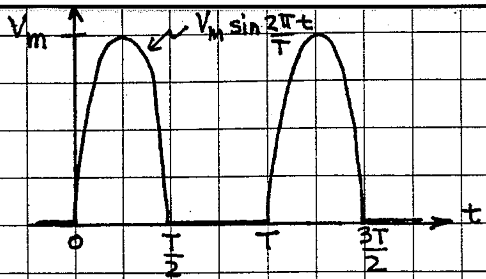


Find Fourier series for $v(t)$.

ans: $v(t) = \frac{V_m}{\pi} + \frac{V_m}{2} \sin \omega_0 t + \frac{2V_m}{\pi} \sum_{k=2,4,6,\ldots}^{\infty} \frac{1}{1-k^2} \cos k\omega_0 t \quad V$

sol'n: $v(t) = a_\nu + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + \sum_{k=1}^{\infty} b_k \cos k\omega_0 t \quad V \qquad \omega_0 \equiv \frac{2\pi}{T}$

Fourier series for $v(t)$. Now we find the coefficients, a_ν, a_k, and b_k.

$$a_\nu = \frac{1}{T}\int_0^T v(t)\,dt \qquad \text{where} \qquad v(t) = \begin{cases} V_m \sin \frac{2\pi}{T} t & 0 \le t \le \frac{T}{2} \\ 0 & \frac{T}{2} \le t \le T \end{cases}$$

Because $v(t)$ is defined to have a different functional form on different intervals, we break $\int_0^T$ into those intervals, $\int_0^{T/2}$ and $\int_{T/2}^{T}$, and we sum the results.

$$a_\nu = \frac{1}{T}\int_0^{T/2} v(t)\,dt + \frac{1}{T}\int_{T/2}^{T} v(t)\,dt$$

$$\text{or} \quad a_\nu = \frac{1}{T}\int_0^{T/2} V_m \sin \frac{2\pi}{T} t \,dt + \frac{1}{T}\int_{T/2}^{T} 0 \,dt$$

Note: We may divide $\int_0^T$ into any number of integrals over subintervals. The only requirement is that those subintervals, placed end-to-end, cover exactly the interval $[0, T]$. The subintervals need not all be the same size, either.

Here, our integrand over the second subinterval, $[T/2, T]$ is zero. Thus, the second integral is zero.

$$\therefore \quad a_\nu = \frac{1}{T}\int_0^{T/2} V_m \sin \frac{2\pi}{T} t \,dt$$

Change variables to simplify notation:

$$\theta \equiv \frac{2\pi}{T}t \qquad \frac{d\theta}{dt} = \frac{2\pi}{T} \Rightarrow d\theta = \frac{2\pi}{T}dt \qquad dt = \frac{T}{2\pi}d\theta$$

$$t = 0 \Rightarrow \theta = 0 \qquad t = \frac{T}{2} \Rightarrow \theta = \frac{2\pi}{2} = \pi$$

Substitute for $\int$ limits: $t=0 \Rightarrow \theta = 0$, $t = \frac{T}{2} \Rightarrow \theta = \pi$

Substitute for $\frac{2\pi}{T}t$: $\sin\frac{2\pi}{T}t \Rightarrow \sin\theta$

Substitute for dt: $dt = \frac{T}{2\pi}d\theta$

$$a_\nu = \frac{1}{T}\int_0^\pi V_m \sin(\theta)\cdot\frac{T}{2\pi}d\theta$$

$$= \frac{V_m}{T}\frac{T}{2\pi}\int_0^\pi \sin\theta\, d\theta$$

$$= \frac{V_m}{2\pi}\int_0^\pi \sin\theta\, d\theta$$

$$= \frac{V_m}{2\pi}(-\cos\theta)\Big|_0^\pi$$

$$= \frac{V_m}{2\pi}(-\cos\pi - -\cos 0)$$

$$= \frac{V_m}{2\pi}\left(-(-1) - -1\right)$$

$$= \frac{2V_m}{2\pi}$$

$$a_\nu = \frac{V_m}{\pi}$$

For a_k and b_k we also break $\int_0^T$ into $\int_0^{T/2} + \int_{T/2}^T$,

and $\int_{T/2}^T = 0$ as it does for a_ν.

$$\therefore \quad a_k = \frac{2}{T}\int_0^{T/2} V_m \sin\frac{2\pi}{T}t\cos\frac{2\pi}{T}t\,dt, \quad b_k = \frac{2}{T}\int_0^{T/2} V_m \sin\frac{2\pi}{T}t\,\sin k\frac{2\pi}{T}t\,dt$$

Change variables, as before:

$$a_k = \frac{2}{T}\int_0^{\pi} V_m \sin\theta \cos k\theta \,\frac{T}{2\pi}\, d\theta \qquad b_k = \frac{2}{T}\int_0^{\pi} V_m \sin\theta \sin k\theta \,\frac{T}{2\pi}\, d\theta$$

We could look up these $\int$'s directly, (see Text p. 1009), but, for the sake of pedagogy, we'll use trigonometric identities to find a_k and complex exponentials to find b_k.

$$\sin\alpha \cos\beta = \frac{\sin(\alpha+\beta)}{2} + \frac{\sin(\alpha-\beta)}{2} \qquad \text{(Text p. 1007)}$$

We identify $\alpha = \theta \quad \beta = k\theta$

$$a_k = \frac{2}{T} V_m \frac{T}{2\pi}\int_0^{\pi} \sin\theta \cos k\theta \, d\theta$$

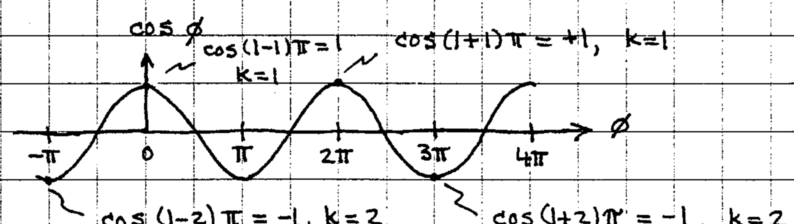
$$= \frac{V_m}{\pi}\int_0^{\pi} \sin\theta \cos k\theta \, d\theta$$

$$= \frac{V_m}{\pi}\int_0^{\pi} \frac{\sin(\theta + k\theta)}{2} + \frac{\sin(\theta - k\theta)}{2} \, d\theta$$

$$= \frac{V_m}{2\pi}\int_0^{\pi} \sin[(1+k)\theta] + \sin[(1-k)\theta] \, d\theta$$

$$= \frac{V_m}{2\pi}\left\{ -\frac{\cos[(1+k)\theta]}{1+k} - \frac{\cos[(1-k)\theta]}{1-k} \right\}\Bigg|_0^{\pi}$$

Note: We should be more careful for $k=1$, since that gives us $\sin(1-k)\theta = \sin 0 = 0$. $\int 0 = 0$, not $\frac{\cos 0}{0}$. We were lucky and got the right answer anyway, but we are not always so lucky...



From the plot of cos( ), we see that:

$$\cos(1+k)\theta\Big|^{\pi} = \begin{cases} +1 & k \text{ odd} \\ -1 & k \text{ even} \end{cases}$$

$$\cos(1-k)\theta\Big|^{\pi} = \begin{cases} +1 & k \text{ odd} \\ -1 & k \text{ even} \end{cases}$$

Also, $\cos(1+k)\cdot 0 = \cos = 1$ and $\cos(1-k)\cdot 0 = 1$.

move minus sign out front

k odd $\quad a_k = -\frac{V_m}{2\pi}\left(\cancel{\frac{+1}{1+k}} \;-\!+\; \cancel{\frac{1}{1+k}} + \cancel{\frac{1}{1-k}} - \cancel{\frac{1}{1-k}}\right) = 0$

k even $\quad a_k = -\frac{V_m}{2\pi}\left(\frac{-1}{1+k} - \frac{1}{1+k} + \frac{-1}{1-k} - \frac{1}{1-k}\right)$

$$= -\frac{V_m}{2\pi}\left(\frac{-2}{1+k} + \frac{-2}{1-k}\right)$$

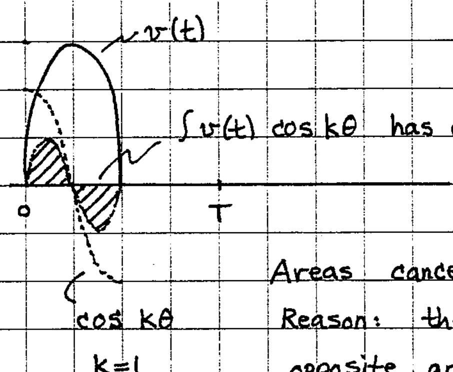
$$= \frac{V_m}{\pi}\left(\frac{1}{1+k} + \frac{1}{1-k}\right)$$

$$= \frac{V_m}{\pi}\left[\frac{1-k}{(1+k)(1-k)} + \frac{1+k}{(1-k)(1+k)}\right]$$

$$a_k = \frac{V_m}{\pi}\,\frac{2}{1-k^2}$$

Summary: $a_k = \begin{cases} 0 & k \text{ odd} \\ \dfrac{2V_m}{\pi(1-k^2)} & k \text{ even} \end{cases}$

Note: The a_k terms are zero because of symmetry:

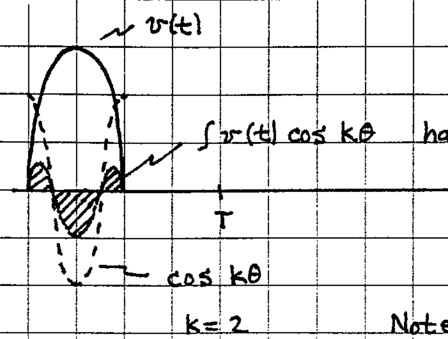


$\int v(t)\cos k\theta$ has equal positive and negative area

$\therefore \int = 0 \Rightarrow a_{k\text{ odd}} = 0$

Areas cancel for all $\cos k\theta$, k odd.
Reason: these cos()'s are equal and opposite around $\frac{T}{4}$, the midpoint of the $v(t)$ lobe.

The a_k for k even are not zero.



has more negative area than positive area when k even,

$\Rightarrow a_{k\,\text{even}} < 0$

Note that $1-k^2 < 0$

$\Rightarrow a_k$ (calculated above) < 0.

We use complex exponentials for b_k calculation (for sake of illustration).

$$b_k = \frac{T}{2\pi}\frac{2}{T}\int_0^{\pi} V_m \sin\theta \sin k\theta \, d\theta \quad \text{from above}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{j2} \qquad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{(for reference)}$$

$$\therefore \quad b_k = \frac{V_m}{\pi}\int_0^{\pi} \sin\theta \sin k\theta \, d\theta$$

$$= \frac{V_m}{\pi}\int_0^{\pi} \frac{e^{j\theta} - e^{-j\theta}}{j2} \cdot \frac{e^{jk\theta} - e^{-jk\theta}}{j2} \, d\theta$$

$$= \frac{V_m}{\pi}\int_0^{\pi} \frac{e^{j(1+k)\theta} + e^{-j(1+k)\theta} - \left(e^{j(1-k)\theta} + e^{-j(1-k)\theta}\right)}{-4} \, d\theta$$

1st · 3rd + 2nd · 4th + 1st · 4th + 2nd · 3rd

Note: We multiply the exponentials in a particular order that gives successive exponents of equal size but opposite signs; e.g. $j(1+k)\theta$, $-j(1+k)\theta$. This procedure allows us to convert back to sin() or cos() later on.

We break b_k into four pieces:

$$b_k = \frac{V_m}{-4\pi}\left(\int_0^{\pi} e^{j(1+k)\theta} \, d\theta + \int_0^{\pi} e^{-j(1+k)\theta} \, d\theta - \int_0^{\pi} e^{j(1-k)\theta} \, d\theta - \int_0^{\pi} e^{-j(1-k)\theta} \, d\theta\right)$$

We integrate complex exponentials the same way we integrate real exponentials:

$$\int e^{cx}\,dx = \frac{e^{cx}}{c} \qquad c \text{ real or complex}$$

$k \neq 1$: $$b_k = -\frac{V_m}{4\pi}\left(\left.\frac{e^{j(1+k)\theta}}{j(1+k)}\right|_0^{\pi} + \left.\frac{e^{-j(1+k)\theta}}{-j(1+k)}\right|_0^{\pi} - \left.\frac{e^{j(1-k)\theta}}{j(1-k)}\right|_0^{\pi} - \left.\frac{e^{-j(1-k)\theta}}{-j(1-k)}\right|_0^{\pi} \right)$$

$$\left. e^{\pm j(1+k)\theta}\right|^{\pi} = e^{j2n\pi} = 1 \quad \text{for } k \text{ odd}$$

$$\left. e^{\pm j(1+k)\theta}\right|^{\pi} = e^{j(2n+1)\pi} = e^{j\pi} = -1 \quad \text{for } k \text{ even}$$

$$\left. e^{\pm j(1-k)\theta}\right|^{\pi} = e^{j2\pi n} = 1 \quad \text{for } k \text{ odd}$$

$$\left. e^{\pm j(1-k)\theta}\right|^{\pi} = e^{j(2n+1)\pi} = e^{j\pi} = -1 \quad \text{for } k \text{ even}$$

$$\left. e^{\pm j(1\pm k)\theta}\right|_0 = e^{j0} = 1 \quad \text{for all } k$$

k odd: $$b_k = -\frac{V_m}{4\pi}\left(\cancel{\frac{1-1}{j(1+k)}} + \cancel{\frac{1-1}{-j(1+k)}} - \cancel{\frac{1-1}{j(1-k)}} - \cancel{\frac{1-1}{-j(1-k)}} \right)$$

$$b_k = 0$$

k even: $$b_k = -\frac{V_m}{4\pi}\left(\frac{-1-\cancel{1}}{j(\cancel{1+k})} + \frac{-1-\cancel{1}}{-j(\cancel{1+k})} - \frac{-1-\cancel{1}}{j(\cancel{1-k})} - \frac{-1-\cancel{1}}{-j(\cancel{1-k})} \right)$$

cancels cancels

$$b_k = 0$$

$\therefore$ All $b_k = 0$ except (possibly) when $k=1$.

Note: We might have forseen this, since $v(t)$ on $[0, \frac{T}{2}]$ acts like an even function for frequencies $k=2$ and higher.

Now for $k=1$.

$$k=1: \quad b_1 = -\frac{V_m}{4\pi}\left(\left.\frac{e^{j2\theta}}{j2}\right|_0^\pi + \left.\frac{e^{-j2\theta}}{-j2}\right|_0^\pi - \int_0^\pi e^{j\cdot 0\cdot\theta}\,d\theta - \int_0^\pi e^{-j\cdot 0\cdot\theta}\,d\theta \right)$$

$$= -\frac{V_m}{4\pi}\left(\frac{\cancel{1-1}}{j2} + \frac{\cancel{1-1}}{-j2} - \int_0^\pi 1\,d\theta - \int_0^\pi 1\,d\theta \right)$$

$$= -\frac{V_m}{4\pi}\left(-\theta\Big|_0^\pi - \theta\Big|_0^\pi \right)$$

$$= +\frac{V_m}{4\pi}\, 2\theta\Big|_0^\pi$$

$$= \frac{V_m}{4\pi}(2\pi - 0)$$

$$= \frac{V_m}{2}$$

Now we put the coefficients, a_v, a_K, b_K into the Fourier series to get:

$$v(t) = \frac{V_m}{\pi} + \left(\sum_{\substack{k \text{ even} \\ k>0}}^{\infty} \frac{2V_m}{\pi(1-k^2)} \cos k\omega_0 t \right) + \frac{V_m}{2} \sin \omega_0 t$$