EX: Verify that the power of a square wave calculated directly from

$$p = \frac{1}{T}\int_0^T v^2(t)dt$$

equals the power calculated from Fourier coefficients.

ANS: Power in both cases is $p = A^2$ where A is the amplitude of the square wave.

SOL'N: Consider a square wave, $v(t)$, of period T that is an odd function (positive with value A from $t = 0$ to $t = T/2$ and value $-A$ from $t = T/2$ to $t = T$).

The square wave, being an odd function, has only sine terms. The square wave, having shift-flip symmetry, has only odd numbered terms.

Direct calculation gives the coefficients for the Fourier series. Because of shift-flip symmetry, we need only integrate from 0 to $T/2$ and double the value.

$$b_k = 2 \cdot \frac{2}{T}\int_0^{T/2} v(t)\sin(2\pi kt/T)\ \ dt$$

$v(t)$ is constant from 0 to $T/2$.

$$b_k = 2 \cdot \frac{2}{T}\int_0^{T/2} A\sin(2\pi kt/T)\ \ dt$$

We may assume any value we desire for T. Here, we will assume $T = 2\pi k$. (Note that we may even use a value for T that changes with k.)

$$b_k = 2 \cdot \frac{2}{2\pi k} A\int_0^{2\pi k/2} \sin(t)\ \ dt = \frac{2A}{\pi k}\left[-\cos(t)\right]\Big|_0^{\pi k}$$

For k odd, $-\cos(\pi k) = 1$, and we obtain our final answer:

$$b_k = \begin{cases} \dfrac{4A}{\pi k} & k \text{ odd } > 0 \\ 0 & \text{otherwise} \end{cases}$$

We calculate the average power in terms of Fourier series coefficients by squaring coefficients and multiplying by 1/2 (since this is the average value of a sinusoid squared). Note that we square and multiply by 1 for the DC offset, (i.e., constant offset), a_v, since this is the average value of 1^2.

$$p = a_v^2 + \tfrac{1}{2}\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right)$$

For our Fourier series we have only odd coefficients for sines:

$$p = \tfrac{1}{2}\sum_{k=1}^{\infty} b_k^2 = \tfrac{1}{2}\sum_{k>0 \text{ odd}}\left(\frac{4A}{\pi k}\right)^2 = \frac{8A^2}{\pi^2}\sum_{k>0 \text{ odd}}\frac{1}{k^2}$$

From Tables of Integrals and Other Mathematical Data by Herbert B. Dwight, we have

$$\sum_{k>0 \text{ odd}}\frac{1}{k^2} = \frac{\pi^2}{8}.$$

Thus, we have $p = A^2$.

Calculating the power directly, we compute the energy in one period and divide by the period:

$$p = \frac{1}{T}\int_0^T v^2(t)\ \ dt$$

The square wave has values $+A$ and $-A$ and a constant, squared value of A^2.

$$p = \frac{1}{T}\int_0^T A^2 \; dt = \frac{1}{T}A^2 T = A^2$$

Thus, the power is the same as calculated from the Fourier series.