23 May 1990 Approximation Theory—
Stone - Weierstrass Theorem examples— Deficient polynomials

Neil E Cotter

ex: $\tilde{\mathcal{F}} = \{ f(x) = \sum_{n=1}^{\infty} a_n x^n \}$ ($a_0 x^0$ term missing) fails S-W thm:

I) Identity $f(x) = 1 \notin \tilde{\mathcal{F}}$ fails all $f(x)$ contain $x^{n \geq 1}$

II) Separability $f(x) = x \in \tilde{\mathcal{F}}$ ✓

III) Closure additive $af(x) + bg(x) = \sum_{n=1}^{\infty} (aa_n + bb_n) x^n \in \tilde{\mathcal{F}}$ ✓

multiplicative $f(x)\,g(x) = \sum_{k=1}^{\infty} \sum_{n+m=k} a_n b_m x^{n+m} \in \tilde{\mathcal{F}}$ ✓

Everything except the identity function works.

ex: $\tilde{\mathcal{F}} = \{ f(x) = ax\,,\ a \in \mathbb{R} \}$ fails S-W thm:

I) Identity $f(x) = 1 \notin \tilde{\mathcal{F}}$ fails

II) Separability $f(x) = x \in \tilde{\mathcal{F}}$ ✓

III Closure additive $f(x) = ax$ $g(x) = bx$ $a_1 f(x) + b_1 g(x) = (a_1 a + b_1 b)\,x \in \tilde{\mathcal{F}}$ ✓

mult. $f(x)\,g(x) = abx^2 \notin \tilde{\mathcal{F}}$ fails

ex: $\tilde{\mathcal{F}} = \{ f(x) = a + bx \}$ fails S-W thm:

I) Identity $f(x) = 1 \in \tilde{\mathcal{F}}$ ✓

II) Sep. $f(x) = x \in \tilde{\mathcal{F}}$ ✓

III) Closure additive $a_1 \overset{a+bx}{f(x)} + b_1 \overset{c+dx}{g(x)} = a_1 a + b_1 c + (a_1 b + b_1 d)x \in \tilde{\mathcal{F}}$ ✓

mult $f(x)\,g(x) = \overset{ac}{\cancel{ab}} + (bc + ad)x + bdx^2 \notin \tilde{\mathcal{F}}$ fails

Mult. closure is often the condition not satisfied.