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notn: $(f, g) \equiv$ inner product of f and g

def: (f_1, f_2) is an inner product $\equiv$ satisfies following

i) $(a_1 f_1 + a_2 f_2, g) = a_1(f_1, g) + a_2(f_2, g)$
ii) $(f_1, f_2) = (f_2, f_1)$
iii) $(f_1, f_1) = \| f_1 \|^2$

where a_1, a_2 are arbitrary constants for the function space containing f_1, f_2, and g,

and $\|f_1\|$ is the norm for the function space.

ex: For a vector space we use the dot product for inner product and $\sqrt{\text{dot product}}$ for the norm.

ex: $V_1 = \begin{bmatrix} 3 \\ 1.2 \end{bmatrix}$ $V_2 = \begin{bmatrix} 0.6 \\ 1 \end{bmatrix}$ $(V_1, V_2) = V_1 \circ V_2 = 3 \cdot 0.6 + 1.2 \cdot 1 = 3.0$

$\|V_1\| = \sqrt{V_1 \circ V_1} = \sqrt{3 \cdot 3 + 1.2 \cdot 1.2} = \sqrt{10.44}$

ex: For function spaces we use the integral of the product of functions for the inner product.

ex: $\mathcal{F}$ = continuous real-valued functions on $[0,1]$.

$$(f_1, f_2) = \int_0^1 f_1(x) f_2(x)\, dx$$

$$(x, 3x^2+1) = \int_0^1 x(3x^2+1)\,dx = \int_0^1 3x^3 + x\, dx = \frac{3x^4}{4}\Big|_0^1 + \frac{x^2}{2}\Big|_0^1$$

$$= \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$$\|f_1\| = \sqrt{\int_0^1 f_1^2\, dx}$$

$$\|x\| = \sqrt{\int_0^1 x^2\, dx} = \sqrt{\frac{x^3}{3}\Big|_0^1} = \sqrt{\frac{1}{3}}$$

ex: Let $f(x) = 1$ for $0 \le x \le 1$ $\|f(x)=1\| = \sqrt{\int_0^1 1\, dx} = \sqrt{x\big|_0^1} = \sqrt{1} = 1$

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ex: $\mathcal{F}$ = continuous real valued functions on $\mathbb{R}^2$

$$(f_1, f_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x_1, x_2)\, f_2(x_1, x_2)\, dx_1\, dx_2$$

$$\| f_1 \| = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1^2(x_1, x_2)\, dx_1\, dx_2}$$

$$\left(f = 1, \frac{1}{x_1^2 x_2^4}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1 \cdot \frac{1}{x_1^2 x_2^4}\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty} \frac{1}{x_1^2} \cdot \left.\frac{-1}{3x_2^3}\right|_{-\infty}^{\infty} dx_1$$

$0 \leftarrow$ undefined, use $2 \cdot \int_0^{\infty}$ instead

$$= \int_{-\infty}^{\infty} \frac{1}{x_1^2}\, 2 \cdot \left(\left.\frac{-1}{3x_2^3}\right|_0^{\infty} \right) dx_1$$

$$= \int_{-\infty}^{\infty} \frac{1}{x_1^2}\, 2 \cdot \infty$$

$$\int_0^{\infty} \frac{1}{1+x^2} = \frac{\pi}{2 \sin \frac{\pi}{2}} = \frac{\pi}{2} \qquad 856.05$$

$$\int_0^{\infty} \frac{1}{(1+x^2)^2} = \frac{\pi}{4} \qquad 856.31$$

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ex: $\mathcal{F}$ = continuous real-valued functions on $\mathbb{R}^2$

$$(f_1, f_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x_1, x_2)\, f_2(x_1, x_2)\, dx_1\, dx_2$$

$$\|f_1\| = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1^2(x_1, x_2)\, dx_1\, dx_2}$$

$$\left(\frac{1}{1+x_1^2}, \frac{5}{(1+x_2^2)^2}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{1+x_1^2}\,\frac{5}{(1+x_2^2)^2}\, dx_1\, dx_2$$

$$= 2\int_0^{\infty} \frac{1}{1+x_1^2}\, dx_1 \cdot 2\int_0^{\infty} \frac{1}{(1+x_2^2)^2}\, dx_2$$

From integral tables by Dwight we have

$$\int_0^{\infty} \frac{1}{1+x_1^2}\, dx_1 = \frac{\pi}{2} \qquad \int_0^{\infty} \frac{1}{(1+x_2^2)^2}\, dx_2 = \frac{\pi}{4}$$

Dwight 856.05 Dwight 856.31

$$\left(\frac{1}{1+x_1^2}, \frac{5}{(1+x_2^2)^2}\right) = 4\cdot\frac{\pi}{2}\cdot\frac{\pi}{4} = \frac{\pi^2}{2}$$