Function Spaces —

Neil E Cotter ~~[illegible]~~ Linear Transforms
9 Apr 1991

Tool: The general form of linear transforms is

$$\mathcal{H}\{f(t)\} \equiv \int_{-\infty}^{\infty} f(\tau)\, k(s,\tau)\, d\tau$$

where $\mathcal{H}\{\cdot\}$ stands for the "$\mathcal{H}$ transform" (for example, $\mathcal{F}\{\ \}$ for Fourier transform and $\mathcal{L}\{\ \}$ for Laplace),

$k(s,\tau)$ is the kernal of the transform (for example, $e^{-j2\pi\tau s}$ for Fourier transform and e^{-st} for Laplace),

and τ is a dummy variable (for example, it could be time or spatial location x).

ex: Fourier $\mathcal{F}\{f(t)\} \equiv F(s) \equiv \int_{-\infty}^{\infty} f(\tau)\, e^{-j2\pi\tau s}\, d\tau$

ex: Laplace $\mathcal{L}\{f(t)\} \equiv F(s) \equiv \int_{0^+}^{\infty} f(\tau)\, e^{-s\tau}\, d\tau$

Note that one-sided Laplace integral lower limit is 0^+ not $-\infty$.

ex: Delta Func $\Delta\{f(t)\} \equiv \int_{-\infty}^{\infty} f(\tau)\, \delta(t-\tau)\, d\tau = f(t)$

Here we use t in the place of s.
This transform gives us $f(t)$ as its result.
Thus, this transform is an identity, and we may think of time domain signals as being "Delta Transformed."

In other words, delta funcs are the basis funcs in the time domain.
$f(\tau)$ tells us how much $\delta(t-\tau)$ is in $f(t)$.

ex: Impulse Response $\mathcal{H}\{f(t)\} \equiv \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$ = system output
System Response

Here $h(t)$ is the system's impulse response.

Function Spaces —

Neil E Cotter ~~Uniqueness of Basis Functions~~, Linear Transforms (cont.)

23 Apr 1991

Tool: The general form of inverse linear transform is

$$f(t) = \mathcal{H}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} k^{(-1)}(s,t)\,F(s)\,ds$$

where $\mathcal{H}^{-1}\{\cdot\}$ stands for the inverse $\mathcal{H}$ transform,

$k^{(-1)}(s,t)$ is the inverse transform kernal satisfying:

$$\delta(t-\tau) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} k^{(-1)}(s,t)\,k(s,\tau)\,ds$$

where $k(s,\tau) = \mathcal{H}\{\delta(t-\tau)\}$ is the kernal of $\mathcal{H}\{\}$,

and $F(s) = \mathcal{H}\{f(t)\}$ is the $\mathcal{H}\{\}$ transform of $f(t)$.

Note: In many cases $k^{(-1)}(s,t) = \dfrac{c}{k(s,\tau)}$ or $\dfrac{c\,\delta(s-s)}{k(s,\tau)}$.

Tool: $F(s)$ indicates how much of $k^{(-1)}(s,t)$ is contained in $f(t)$. Nevertheless, $F(s)$ is computed from an inner product of $f(t)$ and the forward kernal $k(s,t)$.

In other words, $F(s)$ or $\mathcal{H}\{f(t)\}$ tells us how much of the <u>inverse</u> kernal is contained in $f(t)$.

The <u>basis</u> functions after transforming by $\mathcal{H}\{\}$ are the <u>inverse</u> <u>kernals</u>.

Neil E Cotter

Function Spaces — Linear Transforms (cont.)

23 Apr 1991

ex: Linear algebra. Transform to eigenvector basis $\vec{\varphi}_1, \vec{\varphi}_2$

In standard basis $\vec{x} \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

basis vectors $\vec{u}_1, \vec{u}_2$

In eigenvector basis $\vec{x} \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[\begin{bmatrix} \vec{\varphi}_1 \end{bmatrix} \begin{bmatrix} \vec{\varphi}_2 \end{bmatrix}\right] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

eigenvectors $\vec{\varphi}_1, \vec{\varphi}_2$

The linear transform is a summation instead of an integral:

$$\mathcal{H}\{\vec{x}\} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = K\vec{x}$$

Solving for the K (kernal) matrix we have

$$K = \left[\begin{bmatrix} \vec{\varphi}_1 \end{bmatrix} \begin{bmatrix} \vec{\varphi}_2 \end{bmatrix}\right]^{-1}$$

(This is S^{-1} matrix in diagonalization tools)

The inverse transform is seen to be:

$$\mathcal{H}^{-1}\{\vec{c}\} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = K^{-1}\vec{c}$$

where K^{-1} is the inverse kernal matrix

$$K^{-1} = \left[\begin{bmatrix} \vec{\varphi}_1 \end{bmatrix} \begin{bmatrix} \vec{\varphi}_2 \end{bmatrix}\right]$$

(This is S matrix in diagonalization tools)

This example illustrates that $\vec{c}$ tells us how much of each eigenvector is in $\vec{x}$. This follows since the ***inverse*** *kernal is the matrix of eigenvectors, K^{-1}!*

Function Spaces — Linear Transforms (cont.)

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ex: Laplace Transform $\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-s\tau} f(\tau)\, d\tau = F(s)$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{st} F(s)\, ds = f(t)$$

$F(s)$ tells us how much of $\underline{e^{+st}}$ is in $f(t)$.

ex: Fourier Transform $\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{-j2\pi s\tau} f(\tau)\, d\tau = F(s)$

$$\mathcal{F}^{-1}\{F(s)\} = \int_{-\infty}^{\infty} e^{j2\pi st} F(s)\, ds = f(t)$$

$F(s)$ tells us how much of $e^{+j2\pi st}$ is in $f(t)$.

ex: Delta Transform $\Delta\{f(t)\} = \int_{-\infty}^{\infty} \delta(t-\tau) f(\tau)\, d\tau = f(t)$

$$\Delta^{-1}\{f(t)\} = \int_{-\infty}^{\infty} \delta(t-T) f(T)\, dT = f(t)$$

$f(t)$ tells us how much of $\delta(t-T)$ is in $f(t)$.

ex: Integration Transform $\int\{f(t)\} = \int_{-\infty}^{t} f(\tau)\, d\tau = \int_{-\infty}^{\infty} h(t-\tau) f(\tau)\, d\tau = F(t)$

$$\int^{-1}\{F(t)\} = \int_{-\infty}^{\infty} \delta'(t-T)\left[-F(T)\right] dT = f(t)$$

$-\int_{-\infty}^{t} f(\tau)\, d\tau$ tells us how much of $\delta'(t-t)$ is in $f(t)$.

ex: Partial Fractions $PF\{F(s)\} = \frac{1}{2\pi j} \oint_{C_{a+bj}} F(s)\, ds = res(a+bj)$
(F(s) has distinct poles)

contour $\int$ around $s = a+bj$ pole.
contour contains at most one ~~pole~~

$$= \left[s-(a+jb)\right] F(s)\Big|_{s=a+jb}$$

$$PF^{-1}\{res(a+jb)\} = \int_{\substack{\text{Complex}\\ \text{plane}}} \frac{1}{s-(a+jb)}\, res(a+jb)\, d(a+jb)$$

$res(a+jb)$ tells us how much of pole at $s = a+jb$ is in $F(s)$.