Examples of Orthogonal Functions

ex: Legendre polynomials [1]

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
⋮	⋮

Orthogonality:

Domain $D = [-1, 1]$ (possible values for x)

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{mn}$$

$$\equiv \langle P_m(x), P_n(x) \rangle$$

where $\delta_{mn} \equiv \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}$

Note: $w(x) = 1$

Note: Legendre polynomials are orthogonal but not orthonormal. We can make them orthonormal if we multiply them by $\sqrt{\frac{2n+1}{2}}$.

Note: Legendre polynomials solve diff eqn in physics:

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0$$

ref: [1] Wikipedia – Legendre polynomials