Function Spaces

– Orthonormal bases

$\mathcal{G}$ is an orthonormal basis for space $\mathcal{F}$ $\equiv$

i) $\mathcal{G}$ is a set of elements from $\mathcal{F}$

ii) $g_1 \perp g_2$ for any two $g_1, g_2 \in \mathcal{G}$ where $g_1 \neq g_2$

iii) $\|g\|^2 = 1$ for any $g \in \mathcal{G}$

iv) $\mathcal{G}$ is a basis for $\mathcal{F}$

ex: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis for vector space $\mathbb{R}^2$

ex: $\left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix} \right\}$ is an orthonormal basis for vector space $\mathbb{R}^2$ (for any θ)

ex: $\{\sqrt{2}\cos 2\pi n x, \sqrt{2}\sin 2\pi n x : n = 0, \dots, \infty\}$ is an orthonormal basis for continuous real-valued periodic funcs on $[0,1]$

$$\|\sqrt{2}\cos 2\pi n x\|^2 = \int_0^1 \sqrt{2}^2 \cos^2 2\pi n x \, dx = 2 \cdot \frac{1}{2} = 1 \checkmark$$