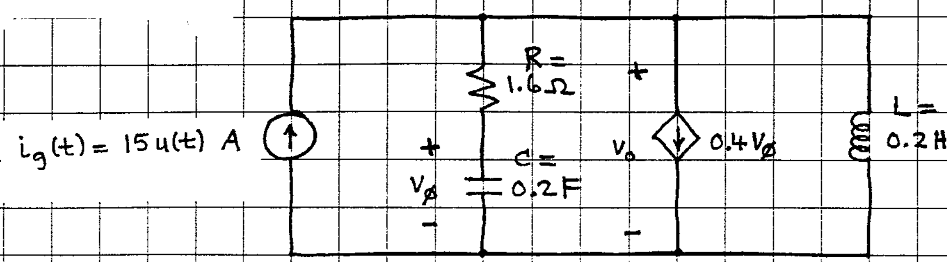
By: Neil E. Cotter

Ref: Nilsson & Riedel, Rev 6, Prob 13.11



Find $v_o(t)$. No energy stored in circuit at $t=0$.

ans: $v_o(t) = [-45te^{-5t} + 24e^{-5t}]u(t)$ V

soln: We apply the general procedure for solving circuit problems in the Laplace domain:

1) Laplace transform each current or voltage source.

Here, $\mathcal{L}\{i_g(t) = 15u(t)\text{ A}\} = 15\cdot\frac{1}{s}\text{ A} \equiv I_g(s)$

The dependent source becomes

$$\mathcal{L}\{0.4v_\phi(t)\} = 0.4V_\phi(s) \quad \text{where } V_\phi(s) \equiv \mathcal{L}\{v_\phi(t)\}$$

Note: It is still a dependent source in the s-domain. It depends on Laplace transformed voltage, $V_\phi(s)$. (We must determine $V_\phi(s)$ at some point below.)

2) Laplace transform each circuit element R, L, or C.

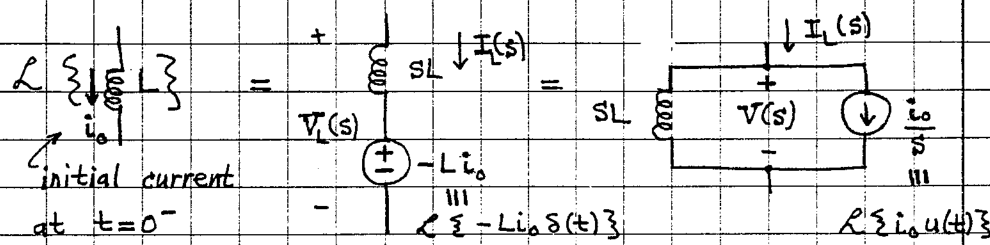
Note: If initial conditions are nonzero for L or C, we may choose whichever of two possible s-domain equivalents is most convenient.

$\mathcal{L}\{R\} = R$ No change.

$v(t) = R\,i(t)$

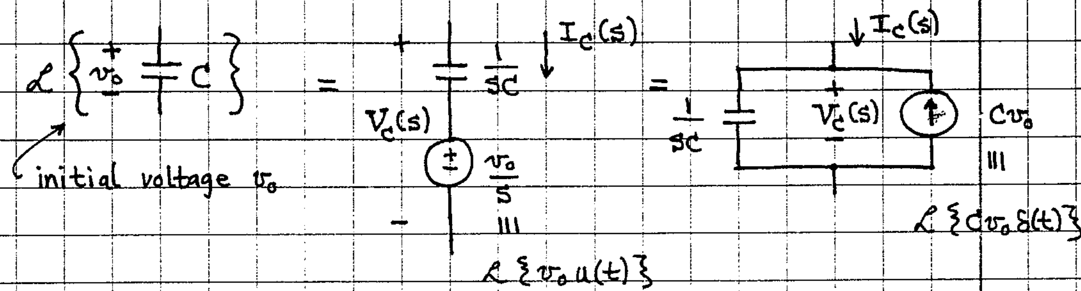
$V(s) = R\,I(s)$

Note: R's are not freq. sensitive ⇒ same in t- and s-domains.



Note: The sources in the s-domain represent signals that create the initial conditions for the L. In the first case, the t-domain equivalent of V-source $-Li_o$ is $-Li_o\delta(t)$. This means that a delta function voltage in series with an L will leave the L with a current of $i_L(t=0^+) = i_o$. In the second case, the ~~t-domain~~ equivalent of I-source i_o/s is $i_o u(t)$. This means that we simply turn on a current source at $t=0$ to carry the initial current on the L. This current source is considered to be inside the L, but we end up treating it like just another circuit element. It is somewhat remarkable that we can handle initial conditions by adding a source that we may treat mathematically as distinct from the L.

Another subtlety is that our s-domain models have zero initial conditions at time $t=0^-$ and become equivalent to the t-domain component at time $t=0^+$ owing to events (such as step or delta functions) occurring at $t=0$. This practice allows us to use superposition when handling initial conditions, (i.e. we just add some sources to our zero-initial-condition circuit).

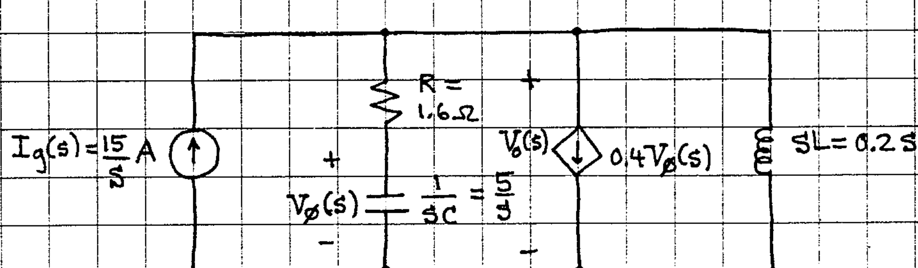


Note: The V-source in the first of the above two s-domain models creates the initial voltage on the C by adding a voltage that steps to v_0 at time $t=0$.

The I-source in the second of the above two s-domain models creates the initial voltage on the C by injecting an appropriate amount of charge on the C via a delta-function current source. The current is infinite for zero duration and delivers a finite charge, (just as the $\delta(t)$ is infinite for zero duration and has a finite area). The voltage we are left with is $v_0 = \frac{1}{C}\int_{0^-}^{0^+} i_C(t)\,dt = \frac{1}{C}\int_{0^-}^{0^+} C v_0 \delta(t)\,dt$

$$= \frac{Cv_0}{C}\int_{0^-}^{0^+} \delta(t)\,dt = \frac{Cv_0}{C}\cdot 1 = v_0. \checkmark$$

Since initial conditions are zero for problem 13.11, our initial condition sources disappear, leaving us with the following s-domain model:



3) In general, we now apply superposition to find $V_o(s)$ as the sum of $V_{oi}(s)$ values for one independent source at at time turned on.

Here, we have only one independent source.

Note: Initial conditions on L's and C's are created by sources that we treat as independent sources. Thus, we effectively turn on one initial condition at a time — or we solve for one input signal (independent source) with initial conditions equal zero — when we apply superposition.

4) Solve the circuit use node-V or mesh-I methods.

Here, we use node-V method:

constraint eq'n: $V_\phi(s) = V_o(s) \cdot \dfrac{1/sC}{1/sC + R}$ V-divider

node-V (sum of currents = 0): $-\dfrac{15}{s} + \dfrac{V_o(s)}{\frac{1}{sC} + R} + 0.4\,V_\phi(s) + \dfrac{V_o(s)}{sL} = 0\text{A}$

$$\text{or } V_o(s)\left[\frac{1}{\frac{1}{sC} + R} + 0.4\,\frac{1/sC}{\frac{1}{sC} + R} + \frac{1}{sL}\right] = \frac{15}{s}$$

$$\text{or } V_o(s)\left[sL + 0.4\,sL/sC + \frac{1}{sC} + R\right] = \frac{15}{\cancel{s}}\,\cancel{s}L\left(\frac{1}{sC} + R\right)$$

$$\text{or } V_o(s)\left[s^2LC + 0.4\,sL + 1 + sRC\right] = 15L\left(1 + sRC\right)$$

We need a coefficient = 1 for highest-order s term

or $$V_o(s)\left[s^2 + \frac{0.4s}{C} + \frac{1}{LC} + s\frac{R}{L}\right] = \frac{15}{\cancel{C}}R\left(\frac{1}{RC} + \frac{s\cancel{RC}}{\cancel{RC}}\right)$$

or $$V_o(s) = \frac{15R\left(s + \frac{1}{RC}\right)}{s^2 + \left(\frac{0.4}{C} + \frac{R}{L}\right)s + \frac{1}{LC}}$$

$$\frac{1}{RC} = \frac{1}{1.6\cdot 0.2} = \frac{5}{1.6} \qquad \frac{0.4}{C} = 5(.04) = 2$$

$$\frac{R}{L} = \frac{1.6}{0.2} = 8 \qquad \frac{1}{LC} = \frac{1}{(0.2)(0.2)} = 25$$

$$15R = 15\cdot 1.6 = 24$$

$$\therefore\ V_o(s) = \frac{24\,(s + 5/1.6)}{s^2 + 10s + 25} = \frac{24s + 75}{(s+5)^2}$$

5) Use partial fraction expansion in s-domain.

$$V_o(s) = \frac{K_1}{(s+5)^2} + \frac{K_2}{s+1}$$

where $$K_1 = V_o(s)\,(s+5)^2\Big|_{s=-5}$$

$$K_2 = \frac{d}{ds}\left[V_o(s)(s+5)^2\right]\Big|_{s=-5}$$

$$K_1 = 24s + 75\Big|_{s=-5} = -45$$

$$K_2 = \frac{d}{ds}(24s+75)\Big|_{s=-5} = 24\Big|_{s=-5} = 24$$

Note: The general formula for K_m with a root repeated n times is:

$$K_m = \frac{1}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left[F(s)\cdot(s+a)^n\right]$$

The $\frac{1}{(m-1)!}$ only comes into play when $m \geq 3$.

6) Use inverse Laplace Transform formula

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^n}\right\} = \frac{t^{n-1}e^{-at}}{(n-1)!}u(t)$$

Note: The $\frac{1}{(n-1)!}$ only comes into play when $n \geq 3$.

Here, we get $v_o(t) = \left[-45te^{-5t} + 24e^{-5t}\right]u(t)$

$$v_o(t) = \left[-45t + 24\right]e^{-5t}u(t).$$