Find $\mathcal{L}\{\ \}$ for a) $g(t) = \int_{0^-}^{t} e^{-ax}\,dx$

b) $g(t) = \int_{0^-}^{t} y\,dy$

c) Check answers by integrating first and then computing $\mathcal{L}\{\ \}$.

ans: a) $\dfrac{1}{s(s+a)}$ b) $\dfrac{1}{s^3}$

sol'n: a) Use operational transform $\mathcal{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \dfrac{F(s)}{s}$ where $F(s) = \mathcal{L}\{f(t)\}$.

Here, $f(t) = e^{-at}$ $\quad \mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}$

$$\therefore \mathcal{L}\left\{\int_{0^-}^{t} e^{-ax}\,dx\right\} = \frac{\frac{1}{s+a}}{s} = \frac{1}{s(s+a)}$$

b) Use same approach as in (a).

Here $f(t) = t$ $\quad \mathcal{L}\{t\} = \dfrac{1}{s^2}$

$$\therefore \mathcal{L}\left\{\int_{0^-}^{t} y\,dy\right\} = \frac{\frac{1}{s^2}}{s} = \frac{1}{s^3}$$

Note: $t = \int_{0^-}^{t} u(x)\,dx$ for $t > 0$ (see Note 1 at end of sol'n)

$$\therefore \mathcal{L}\{t\} = \frac{\mathcal{L}\{u(t)\}}{s} = \frac{1}{s^2}$$

$u(t) = \int_{0^-}^{t} \delta(x)\,dx$ for $t > 0$

$$\therefore \mathcal{L}\{u(t)\} = \frac{\mathcal{L}\{\delta(t)\}}{s} = \frac{1}{s}$$

and $\mathcal{L}\{\delta(t)\} = 1$ is starting point

successive $\int_{0^-}^{t} dt$'s add $\frac{1}{s}$ each time

c) $$\int_{0^-}^{t} e^{-ax}\,dx = \left.\frac{e^{-ax}}{-a}\right|_{0^-}^{t} = \frac{e^{-at}}{-a} - \frac{e^{0^-}}{-a} = \frac{1}{a}\left(1-e^{-at}\right)$$

$$\mathcal{L}\left\{\frac{1}{a}\left(1-e^{-at}\right)\right\} = \frac{1}{a}\mathcal{L}\{1\} - \frac{1}{a}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{a}\mathcal{L}\{u(t)\} - \frac{1}{a}\frac{1}{s+a}$$

$$= \frac{1}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right) = \frac{1}{\cancel{a}\,1}\,\frac{\cancel{s}\cancelto{1}{+a} - \cancel{s}}{s(s+a)}$$

$$= \frac{1}{s(s+a)} \quad \checkmark$$

$$\int_{0^-}^{t} y\,dx = \left.\frac{y^2}{2}\right|_{0}^{t} = \frac{t^2}{2} - 0 = \frac{t^2}{2}$$

$$\boxed{\text{Note: } \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \text{(not in book's tables)}}$$

$$\therefore \mathcal{L}\left\{\frac{t^2}{2}\right\} = \left.\frac{1}{2}\,\frac{2!}{s^{n+1}}\right|_{n=2} = \frac{1}{s^3} \quad \checkmark$$

Note 1: Actually, $tu(t) = \int_{0^-}^{t} u(x)\,dx$ is a better definition. $\mathcal{L}\{tu(t)\} = \mathcal{L}\{t\}$, however.

The exact definition we use here only makes a difference if we want to take derivatives.