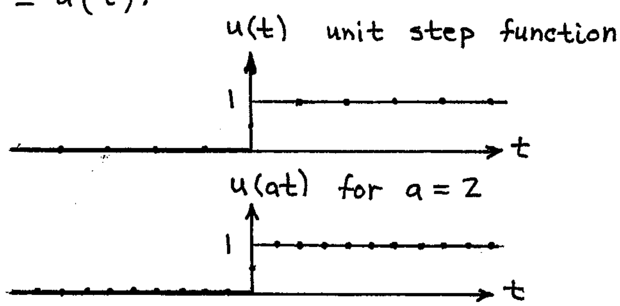
EX: Find the Laplace transform of the following waveform: (assume $a>0$)

$$f(t)=\frac{1}{t}\int_0^t u(at)dt$$

soln: It is instructive to consider this problem in the time domain before solving it with transform identities (which is usually the preferred approach).

If we plot $u(at)$, we discover that $u(at) = u(t)$:



Thus, we have $\mathcal{L}\{u(at)\} = \mathcal{L}\{u(t)\} = \frac{1}{s}$.

If we use Laplace transforms directly on $u(at)$, we apply the identity for scaling time:

$$\mathcal{L}\{f(at) \text{ for } a>0\} = \frac{1}{a}\mathcal{L}\{f(t)\}\Big|_{\frac{s}{a} \text{ replaces } s}$$

$$\mathcal{L}\{u(at)\} = \frac{1}{a}\mathcal{L}\{u(t)\}\Big|_{\frac{s}{a} \text{ replace } s}$$

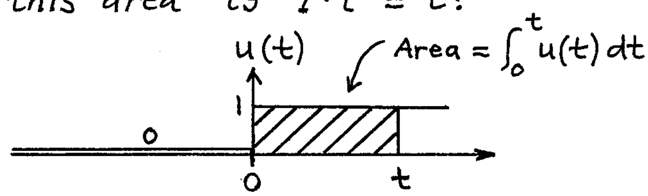
$$= \frac{1}{a}\frac{1}{s}\Big|_{\frac{s}{a} \text{ replaces } s}$$

$$= \frac{1}{a}\frac{1}{s/a} = \frac{1}{s} \checkmark \text{(as before)}$$

In the time domain, we have

$$\int_0^t u(at)\,dt = \int_0^t u(t)\,dt = t$$

since the integral is the area under the unit step function from 0 to t, and this area is $1 \cdot t = t$:



Thus, $\mathcal{L}\left\{\int_0^t u(at)\,dt\right\} = \mathcal{L}\{t\} = \dfrac{1}{s^2}$ (from table).

If we use the Laplace transform directly, we apply the identity for integration in the time domain:

$$\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\left\{\int_0^t u(at)\,dt\right\} = \frac{1}{s}\mathcal{L}\{u(at)\} = \frac{1}{s}\cdot\frac{1}{s} \quad \checkmark$$

In the time domain, we have

$$\frac{1}{t}\int_0^t u(at)\,dt = \frac{1}{t}\cdot t = 1.$$

$$\therefore \mathcal{L}\left\{\frac{1}{t}\int_0^t u(at)\,dt\right\} = \mathcal{L}\{1\} = \mathcal{L}\{u(t)\} = \frac{1}{s}$$

If we use the Laplace transform directly, we apply the identity for multiplication by $\frac{1}{t}$:

$$\mathcal{L}\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \mathcal{L}\{f(t)\}\,ds'$$

Note: we use variable of integration s' instead of s to avoid confusion with the lower limit of the integral. s' is a frequency variable like s; s' does <u>not</u> mean ds/dt.

$$\mathcal{L}\left\{\frac{1}{t}\int_0^t u(at)\,dt\right\} = \int_s^\infty \mathcal{L}\left\{\int_0^t u(at)\,dt\right\} ds'$$

$$= \int_s^\infty \frac{1}{s'^2}\,ds'$$

$$= \left.\frac{-1}{s'}\right|_{s'=s}^{s'=\infty}$$

$$= -\cancel{\frac{1}{\infty}}^{\,0} - -\frac{1}{s}$$

$$\mathcal{L}\left\{\frac{1}{t}\int_0^t u(at)\,dt\right\} = \frac{1}{s} \quad ✓$$