EX: Find the following Laplace transform:

$$\mathcal{L}\left\{\int_0^t e^{-6\tau} u(6\tau) d\tau\right\}$$

SOL'N: Apply the scaling identity with $a = 6$ to $f(t) = e^{-t}u(t)$:

$$\mathcal{L}\{f(at)\} = \frac{1}{a}\mathcal{L}\{f(t)\}\Big|_{\text{replace } s \text{ with } \frac{s}{a}}$$

The result:

$$\frac{1}{6}\mathcal{L}\left\{e^{-6t}u(6t)\right\} = \frac{1}{6}\mathcal{L}\left\{e^{-t}u(t)\right\}\Big|_{\text{replace } s \text{ with } \frac{s}{6}}$$

$$\frac{1}{6}\mathcal{L}\left\{e^{-6t}u(6t)\right\} = \frac{1}{6}\frac{1}{(s+1)}\Big|_{\text{replace } s \text{ with } \frac{s}{6}}$$

The result we obtain when we simplify is the same as we would have if our original expression were written with $u(t)$ in place of $u(6t)$. This makes sense since scaling of t leaves $u()$ unchanged.

$$\mathcal{L}\left\{e^{-6t}u(6t)\right\} = \frac{1}{6}\frac{1}{\left(\frac{s}{6}+1\right)} = \frac{1}{s+6}$$

To account for the integral, we multiply the previous answer by $1/s$:

$$\mathcal{L}\left\{\int_0^t e^{-6\tau} u(6\tau) d\tau\right\} = \frac{1}{s}\cdot\frac{1}{s+6} = \frac{1}{s(s+6)}$$

This problem is simple enough that we may easily check our answer by computing the integral before taking the Laplace transform:

$$\mathcal{L}\left\{\int_0^t e^{-6\tau} u(6\tau) d\tau\right\} = \mathcal{L}\left\{\frac{e^{-6t}}{-6}\Bigg|_0^t\right\} = \mathcal{L}\left\{\frac{1}{-6}(e^{-6t}-1)\right\}$$

or

$$\mathcal{L}\left\{\int_0^t e^{-6\tau} u(6\tau) d\tau\right\} = \frac{1}{6}\left(\frac{1}{s}-\frac{1}{s+6}\right) = \frac{1}{s(s+6)} \quad \surd$$