EX: Find the inverse Laplace transform for the following expression:

$$F(s) = \frac{4s+10}{s^2+8s+25}$$

SOL'N: Our first step is, as always, to find the roots of the denominator. This amounts to solving a quadratic equation. In the general case we have two roots that may be real or complex:

$$s^2 + b's + c' = \left(s + \frac{b'}{2} + \sqrt{\left(\frac{b'}{2}\right)^2 + c'}\right)\left(s + \frac{b'}{2} - \sqrt{\left(\frac{b'}{2}\right)^2 + c'}\right)$$

If the term inside the radical is negative, the roots are complex, and we write the roots in the following form:

$$s^2 + 2as + a^2 + \omega^2 = (s + a + j\omega)(s + a - j\omega)$$

or

$$s^2 + 2as + a^2 + \omega^2 = (s + a)^2 + \omega^2$$

In these forms, we observe that a is half the value of the s coefficient. Having found a, we find ω as the square root of the constant term minus a^2.

For the problem at hand, we have

$$s^2 + 8s + 25 = (s + 4 + j3)(s + 4 - j3) = (s + 4)^2 + 3^2$$

One way to proceed is to use standard partial fraction techniques:

$$F(s) = \frac{K}{s + 4 + j3} + \frac{K^*}{s + 4 - j3}$$

Note that we always have complex conjugate values for the partial fraction coefficients, meaning we only have to find one of the coefficients.

$$K = (s + 4 + j3)F(s)\Big|_{s=-4-j3} = \frac{4s+10}{s+4-j3}\Bigg|_{s=-4-j3}$$

The denominator in this expression will always be twice the imaginary part of the remaining root.

$$K = \frac{4(-4-j3)+10}{-2 \cdot j3} = \frac{-6-j12}{-j6} = \frac{j(-1-j2)}{j \cdot (-j)} = 2 - j$$

The partial fraction expression for $F(s)$ may be reduced to terms corresponding to decaying cosine and sine terms in the time domain:

$$\mathcal{L}\{e^{-at}\cos(\omega t)\} = \frac{s+a}{(s+a)^2+\omega^2}$$

$$\mathcal{L}\{e^{-at}\sin(\omega t)\} = \frac{\omega}{(s+a)^2+\omega^2}$$

Here, we solve the general case:

$$F(s) = \frac{a'+jb'}{s+a+j\omega} + \frac{a'-jb'}{s+a-j\omega}$$

We use a common denominator:

$$F(s) = \frac{(a'+jb')(s+a-j\omega)+(a'-jb')(s+a+j\omega)}{(s+a)^2+\omega^2}$$

After simplifications, we have a form in which we can identify the decaying cosine and sine terms:

$$F(s) = \frac{2a'(s+a)+2b'\omega}{(s+a)^2+\omega^2}$$

The inverse transform reveals that twice the real part of the partial fraction coefficient, K, is the magnitude of the decaying cosine term, and twice the imaginary part of the partial fraction coefficient, K, is the magnitude of the decaying sine term.

$$f(t) = 2a'e^{-at}\cos(\omega t) + 2b'e^{-at}\sin(\omega t)$$

For the problem at hand, we have $K = 2 - j$:

$$f(t) = 2(2)e^{-at}\cos(\omega t) + 2(-1)e^{-at}\sin(\omega t)$$

or

$$f(t) = 4e^{-4t}\cos(3t) - 2e^{-4t}\sin(3t)$$

An alternate approach is to bypass the partial fraction coefficient calculation and instead start with the expression written as a sum of terms for the decaying cosine and sine:

$$F(s) = \frac{2a'(s+a)+2b'\omega}{(s+a)^2+\omega^2}$$

We solve for the values of a' and b' that make the numerator of this expression equal the numerator of our $F(s)$.

$$2a'(s+a)+2b'\omega = 4s+10$$

Equating coefficients of powers of s, we have two equations in two unknowns:

$$2a's = 4s$$

$$2a'a + 2b'\omega = 10$$

Given $a = 4$ and $\omega = 3$ we find $a' = 2$ and $b' = -1$. We then apply a result derived above:

$$f(t) = 2a'e^{-at}\cos(\omega t) + 2b'e^{-at}\sin(\omega t)$$

We obtain the answer found earlier:

$$f(t) = 4e^{-4t}\cos(3t) - 2e^{-4t}\sin(3t)$$