EX: Find the inverse Laplace transform for the following expression:

$$F(s) = \frac{7s^2 + 14s + 9}{(s+1)^3}$$

SOL'N: We have repeated roots which necessitates the use of a different method than we would use for single roots. We write a partial fraction term for each power of the root term:

$$F(s) = \frac{K_1}{(s+1)^3} + \frac{K_2}{(s+1)^2} + \frac{K_3}{s+1}$$

To find K_1, we multiply by the highest power of the root term and evaluate the expression when the root term equals zero:

$$K_1 = (s+1)^3 F(s)\big|_{s+1=0}$$

NOTE: To see why this works, consider what is happening when F(s) is written in terms of partial fractions:

$$(s+1)^3 F(s)\big|_{s+1=0} = \frac{K_1(s+1)^3}{(s+1)^3} + \frac{K_2(s+1)^3}{(s+1)^2} + \frac{K_3(s+1)^3}{s+1}\bigg|_{s+1=0}$$

After canceling terms, we find that K_2 and K_3 are multiplied by zero:

$$(s+1)^3 F(s)\big|_{s+1=0} = K_1 + K_2(s+1) + K_3(s+1)^2\big|_{s+1=0}$$

For the problem at hand, we calculate K_1 as follows:

$$K_1 = (s+1)^3 F(s)\big|_{s+1=0} = 7s^2 + 14s + 9\big|_{s+1=0}$$

or

$$K_1 = 7s^2 + 14s + 9\big|_{s=-1} = 7 - 14 + 9 = 2$$

If we try to find K_2 by multiplying $F(s)$ by $(s + 1)^2$ and evaluating at $s + 1 = 0$, we find that we have a divide by zero in the first term containing K_1. Thus, we must find a new approach that will yield the value of K_2 (and K_3).

One method that works is to take the derivative (with respect to s) of $(s + 1)^3F(s)$ and evaluate that function where the root equals zero:

$$K_2 = \left\{\frac{d}{ds}\left[(s+1)^3 F(s)\right]\right\}\bigg|_{s+1=0}$$

NOTE: We take the derivative before we evaluate at the root value. Otherwise, we are differentiating a constant and always get zero as our answer.

NOTE: To see why this works, again consider what is happening when F(s) is written in terms of partial fractions:

$$\frac{d}{ds}\left[(s+1)^3 F(s)\right] = \frac{d}{ds}\left[K_1 + K_2(s+1) + K_3(s+1)^2\right]$$

Because the derivative of K_1 is zero, it is eliminated. K_2 appears without a multiplier, and K_3 is multiplied by the root term, which will be zero when we evaluate at the value of the root.

$$\frac{d}{ds}\left[(s+1)^3 F(s)\right] = K_2 + 2K_3(s+1)$$

Note that K_3 is multiplied by 2 rather than 1. This will affect the calculation of K_3, below.

For the problem at hand, we calculate K_2 as follows:

$$K_2 = \frac{d}{ds}(s+1)^3 F(s)\Big|_{s+1=0} = \frac{d}{ds} 7s^2 + 14s + 9\Big|_{s+1=0}$$

or

$$K_2 = 14s + 14\Big|_{s+1=0} = -14 + 14 = 0$$

To find K_3, we differentiate again and evaluate where the root is zero. We must divide by 2, however, because the first time we differentiate we get $2K_3(s+1)$.

$$K_3 = \frac{1}{2}\frac{d^2}{ds^2}(s+1)^3 F(s)\Big|_{s+1=0}$$

NOTE: In general, if we start with a root of order n, we will have to divide by $(m-1)!$ when finding K_m.

$$K_m = \frac{1}{(m-1)!}\frac{d^{(m-1)}}{ds^{(m-1)}}(s+a)^n F(s)\Big|_{s+a=0}$$

For the problem at hand, we calculate K_3 as follows:

$$K_3 = \frac{1}{2}\frac{d^2}{ds^2}(s+1)^3 F(s)\Big|_{s+1=0} = \frac{1}{2}\frac{d}{ds} 14s + 14\Big|_{s+1=0}$$

or

$$K_3 = \frac{1}{2} 14\Big|_{s+1=0} = 7$$

Again, we note that we must take all derivatives before evaluating the expression.

An alternative to the above approach is to find K_1 in the same manner but find K_2 and K_3 by evaluating $(s + 1)^3 F(s)$ at convenient values of s. The convenient values chosen are up to the user, but choosing values of s that yield root terms equal to plus or minus one are convenient here:

$$(s+1)^3 F(s)\Big|_{s+1=1} = K_1 + K_2(s+1) + K_3(s+1)^2\Big|_{s+1=1}$$

$$(s+1)^3 F(s)\Big|_{s+1=1} = K_1 + K_2 + K_3$$

and

$$(s+1)^3 F(s)\Big|_{s+1=-1} = K_1 + K_2(s+1) + K_3(s+1)^2\Big|_{s+1=-1}$$

$$(s+1)^3 F(s)\Big|_{s+1=-1} = K_1 - K_2 + K_3$$

For the problem at hand, we calculate values as follows:

$$(s+1)^3 F(s)\Big|_{s+1=1} = 7s^2 + 14s + 9\Big|_{s=0} = 9$$

or

$$K_1 + K_2 + K_3 = 9$$

and

$$(s+1)^3 F(s)\Big|_{s+1=-1} = 7s^2 + 14s + 9\Big|_{s=-2} = 9$$

or

$$K_1 - K_2 + K_3 = 9$$

From the two equations, we see that $K_2 = 0$, and using $K_1 = 2$ we see that $K_3 = 7$.

Now we are ready to find the inverse Laplace transform. The following identity is helpful:

$$L^{-1}\left\{\frac{1}{(s+a)^n}\right\} = \frac{t^{n-1}}{(n-1)!}e^{-at}$$

For the problem at hand, we have the following partial fraction expression:

$$F(s) = \frac{2}{(s+1)^3} + \frac{0}{(s+1)^2} + \frac{7}{s+1}$$

Applying the identity yields our final time-domain answer:

$$f(t) = \frac{t^2}{2}e^{-t} + 7e^{-t}$$