30 Mar 1990 Linear Algebra — Rotation Matrices

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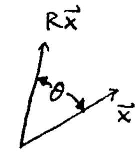
R is a rotation matrix if it rotates vectors through angle θ without changing vector length.

ex: 2-Dim case



$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

The eigenvalues of R must have absolute value = 1 since lengths not changed.

$$0 = |R - \lambda I| = \left| \begin{bmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{bmatrix} \right| \qquad |\text{matrix}| = \text{determinan}$$

$$= (\cos\theta - \lambda)^2 + \sin^2\theta$$

$$= \cos^2\theta - 2\cos\theta\lambda + \lambda^2 + \sin^2\theta$$

$$= 1 - 2\cos\theta\,\lambda + \lambda^2 \qquad \cos^2 + \sin^2 = 1$$

Now $(\lambda - a)(\lambda - b) = \lambda^2 - (a+b)\lambda + ab = 0$

const term = product of λ's

$\therefore$ we see that $|\lambda_1||\lambda_2| = 1$, and $\det R \equiv |R| = |\lambda_1||\lambda_2| = 1$

With more work on quadratic eq'n for λ we get $|\lambda_1| = |\lambda_2| = 1$

ex: $\theta = 30°$ $\quad \cos\theta = \frac{\sqrt{3}}{2}$ $\quad \sin\theta = \frac{1}{2}$

$$1 - \cancel{2}\frac{\sqrt{3}}{\cancel{2}}\lambda + \lambda^2 = 0 \qquad \lambda = \frac{-\sqrt{3} \pm \sqrt{\sqrt{3}^2 - 4}}{2} = \frac{-\sqrt{3} \pm j}{2}$$

$$|\lambda_1| = \left| \frac{-\sqrt{3} + j}{2} \right| = \sqrt{\frac{3+1}{4}} = 1$$

Note: eigvecs complex, (must be, since all real vecs get rotated 30°).