Ex: Derive the state-space equations (i.e., first derivatives of state variables i, ω, and θ on the left) for a single-coil motor with the following magnetic flux, ψ, versus rotor angle θ, and current, *i*:

$$\psi = \psi_o + L_1 \sin(\theta) i$$

sol'n: Given $\psi = \psi_o + L_1 \sin(\theta) i$,

we substitute into the v eq'n:

$v = Ri + \frac{d\psi}{dt}$ and solve for $\frac{di}{dt}$.

Then we obtain power as vi and look for power available as mechanical power, p_{mech}.

Then we say $p_{mech} = \tau_{mech}\,\omega$

or $\tau_{mech} = \frac{p_{mech}}{\omega}$.

Then we say $J\frac{d\omega}{dt} = \tau_{mech} - \tau_{LF}$

or $\frac{d\omega}{dt} = \frac{1}{J}(\tau_{mech} - \tau_{LF})$.

Finally, we add $\frac{d\theta}{dt} = \omega$.

Now for the problem at hand:

Write ψ as $\psi = \psi_o + L(\theta)\, i$

where $L(\theta) = L_1 \sin(\theta)$.

We have $\frac{d\psi}{dt} = \frac{\partial L(\theta)}{\partial \theta} \frac{d\theta}{dt} i + L(\theta) \frac{di}{dt}$

or $\frac{d\psi}{dt} = \frac{\partial L(\theta)}{\partial \theta} \omega i + L(\theta) \frac{di}{dt}$

where $\frac{\partial L(\theta)}{\partial \theta} = \frac{\partial}{\partial \theta} L_1 \sin(\theta)$

$\qquad'' = L_1 \cos(\theta)$

$$\therefore v = Ri + L_1 \cos(\theta) \omega i + \underbrace{L_1 \sin(\theta)}_{\equiv L(\theta)} \frac{di}{dt}$$

$$\boxed{\frac{di}{dt} = \frac{1}{L(\theta)} \left[v - Ri - L_1 \cos(\theta) \omega i \right]}$$

Now we compute power $= vi$:

split into two pieces

$$vi = \underbrace{Ri^2}_{\substack{\text{heat lost in} \\ \text{R of coil}}} + \underbrace{\frac{1}{2} \frac{\partial L(\theta)}{\partial \theta} \omega i^2}_{P_{mech}} + \underbrace{L(\theta) \frac{di}{dt} \cdot i + \frac{1}{2} \frac{\partial L(\theta)}{\partial \theta} \omega i^2}_{\text{stored pwr}}$$

heat lost in R of coil

Pmech comes from change in L with position; similar to $\frac{d}{dt} \frac{1}{2} L i^2$ when i constant and L changing

stored pwr similar to $\frac{d}{dt} \frac{1}{2} L i^2$ when L is constant. This pwr is returned to circuit. $pwr = \frac{d}{dt} \frac{1}{2} L(\theta) i^2$

$$P_{mech} = \frac{1}{2}\frac{\partial L(\theta)}{\partial \theta}\omega i^2$$

$$" = \frac{1}{2}L_1 \cos(\theta)\,\omega\, i^2$$

$$" = \tau_{mech}\,\omega$$

$$\therefore \tau_{mech} = \frac{1}{2}L_1\cos(\theta)\, i^2$$

Now use $J\frac{d\omega}{dt} = \tau_{mech} - \tau_{LF}$.

In this problem, we assume $\tau_{LF} = 0$.

$$\therefore J\frac{d\omega}{dt} = \frac{1}{2}L_1\cos(\theta)\, i^2$$

or

$$\boxed{\frac{d\omega}{dt} = \frac{1}{2}\frac{L_1}{J}\cos(\theta)\, i^2}$$

Last, we always have

$$\boxed{\frac{d\theta}{dt} = \omega}$$