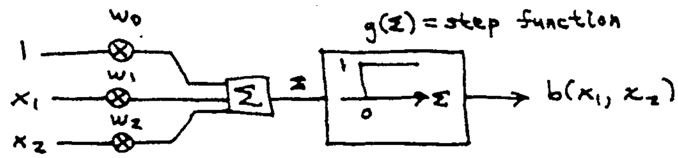
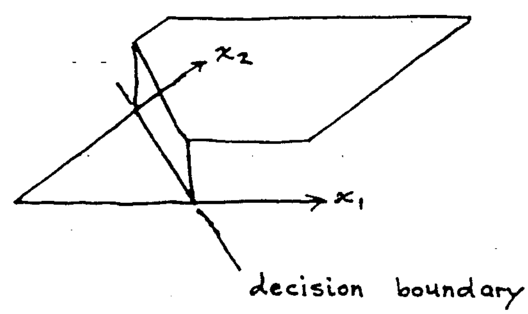
Neil E Cotter
6 June 1994



$$b(x_1, x_2) = g(\Sigma\, w_i x_i) = g(\vec{w}_+ \cdot \vec{x}_+)$$

$$\vec{w}_+ \equiv (w_0, w_1, w_2) \qquad \vec{x}_+ \equiv (1, x_1, x_2)$$

decision boundary at $\vec{w}_+ \circ \vec{x}_+ = 0$

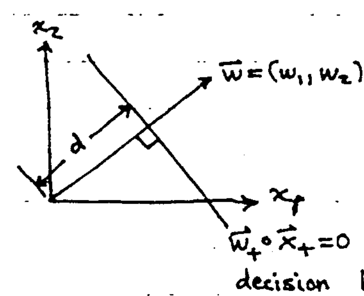


claim: $\vec{w} = (w_1, w_2)$ is $\perp$ (perpendicular to) decision boundary

and distance from origin to decision boundary is

$$d = \min_{\substack{x \text{ on decision} \\ \text{boundary}}} |\vec{x}| = -\operatorname{sgn}(w_0)\frac{|w_0|}{|\vec{w}|} = \frac{-w_0}{|\vec{w}|}$$

pf: top view



First, we prove $\vec{w} \perp$ decision boundary.

Let $\vec{x}_0$ be any point on decision boundary.

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pf (cont.): We have $\vec{w}_+ \circ \vec{x}_{0+} = 0$ or $w_0 + \vec{w} \circ \vec{x}_0 = 0$

Let $\vec{x}_1$ be any other point on the decision boundary.

Define $\vec{v} = \vec{x}_1 - \vec{x}_0$.

We have
$$\begin{aligned} \vec{w}_+ \circ \vec{x}_{1+} = 0 &= w_0 + \vec{w} \circ (\vec{x}_0 + \vec{v}) \\ &= w_0 + \vec{w} \circ \vec{x}_0 + \vec{w} \circ \vec{v} \\ &= 0 \qquad\qquad + \vec{w} \circ \vec{v} \end{aligned}$$

Thus $\vec{w} \circ \vec{v} = 0$ and $\vec{v} \perp \vec{w}$.

This means $\vec{x}_1$ lies on a line $\perp \vec{w}$.

pf (of formula for d): Since we know $\vec{w} \perp$ decision boundary we know the min distance to the boundary lies in the direction of $\vec{w}$.

Then d = multiplier of unit vector in $\vec{w}$ direction that yields a solution to $\vec{w}_+ \circ \vec{x}_{0+} = 0$.

That is $w_0 + \vec{w} \circ d \dfrac{\vec{w}}{|\vec{w}|} = 0$.

$$d \, |\vec{w}| = -w_0$$

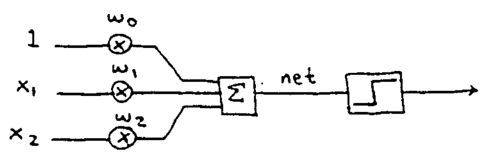
$$d = \frac{-w_0}{|\vec{w}|}.$$

note: The above claim also holds in higher dimensions.

4 Apr 1990 Perceptrons – Decision Boundary

Neil E Cotter

Consider perceptron which uses w_0 as threshold:

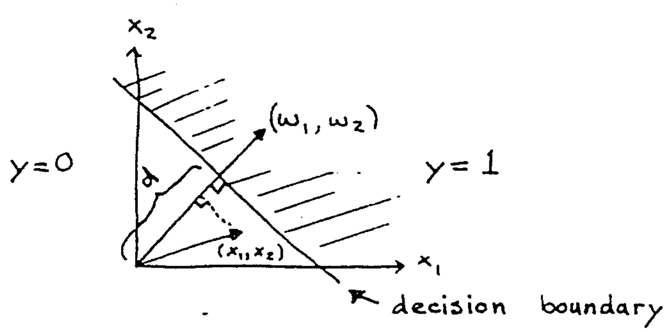


$$y = \begin{cases} 0 & \text{net} < 0 \\ 1 & \text{net} \geq 1 \end{cases}$$

Then the decision boundary has these properties:

1) Vector (w_1, w_2) is $\perp$ boundary

2) shortest distance from boundary to origin is $d = \dfrac{-w_0}{|(w_1,w_2)|}$



pf: boundary is where $(w_0, w_1, w_2) \circ (1, x_1, x_2) = 0$

or $w_0 + (w_1,w_2) \circ (x_1, x_2) = 0$

$$(w_1,w_2) \circ (x_1,x_2) = -w_0$$

$$\underbrace{\frac{(w_1,w_2) \circ (x_1,x_2)}{|(w_1,w_2)|}} = \frac{-w_0}{|(w_1,w_2)|}$$

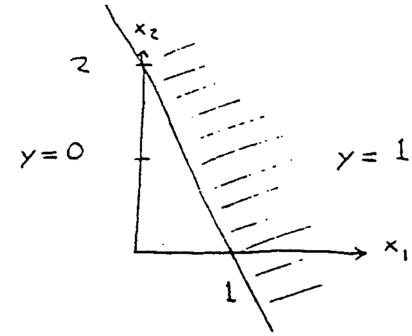
left hand side is projection of (x_1,x_2) onto (w_1,w_2) vector $= p((x_1,x_2),(w_1,w_2))$

$\therefore$ boundary consists of endpts of all (x_1,x_2) vecs whose projection onto (w_1,w_2) has length $\dfrac{-w_0}{|(w_1,w_2)|} \equiv d$. Since we find projection by dropping a perpendicular from (x_1,x_2) to (w_1,w_2) all pts. on boundary are on line $\perp$ (w_1,w_2).

4_Apr 1990 Perceptron - Decision Boundary (cont.)

Neil E Cotter

ex: find (w_0, w_1, w_2) for following decision boundary:



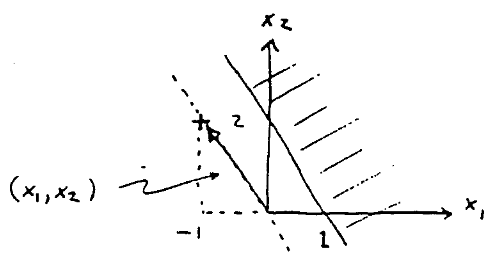
soln (not unique): 1st find $(w_1, w_2) \perp (x_1, x_2)$

$$(w_1, w_2) \perp (x_1, x_2) \Rightarrow (w_1, w_2) \circ (x_1, x_2) = 0$$

$$\text{or} \quad w_1 x_1 + w_2 x_2 = 0$$

Q. But what is (x_1, x_2)?

A. It is vector from origin and $\parallel$ (parallel) to decision boundar[y]



Use $(x_1, x_2) = (-1, 2)$. Length of this vec doesn't matter,
(is $\parallel$ decision boundary) but <u>direction</u> <u>does</u> matter.

So $w_1 x_1 + w_2 x_2 = w_1(-1) + w_2(2) = 0$

One eq'n, two unknowns; choose any w_1 desired.

For 2-D case can use trick:

$$(-x_2, x_1) \perp (x_1, x_2) \quad \text{or} \quad (x_2, -x_1) \perp (x_1, x_2)$$

So use $(w_1, w_2) = (x_2, -x_1) = (2, 1)$.

$$(w_1, w_2) = (2, 1)$$

Note: $(w_1, w_2) = (-2, -1)$ flips the sides for $y=0$ and $y=1$.

5 Apr 1990 Perceptron — Decision Boundary (cont.)

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To find threshold w_0 we can use any point on the decision boundary.

(1,0) and (0,2) are on the boundary. Use (1,0).

$$w_0 + (w_1, w_2) \circ (x_1, x_2) = 0$$

$$(2, 1) \circ (1, 0) = 0$$

$$w_0 + 2 \cdot 1 + 1 \cdot 0 = 0$$

$$w_0 + 2 = 0$$

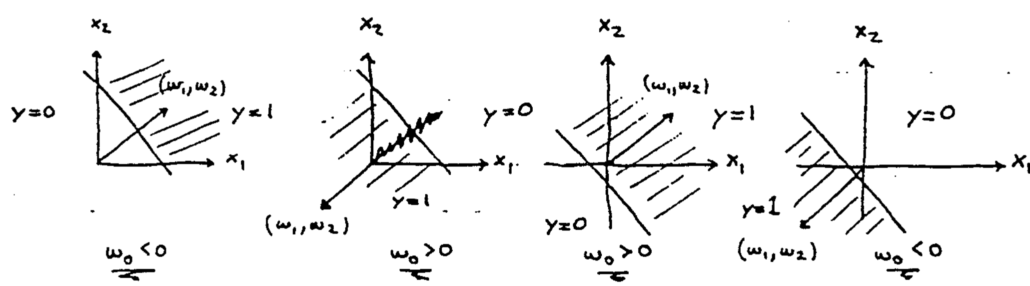
$$w_0 = -2$$

We should also verify that the region for $y = 1$ is above the line rather than below it. Pick any point, say (3,0), in the $y=1$ region.

$$w_0 + (w_1, w_2) \circ (3,0) = -2 + 2 \cdot 3 + 1 \cdot 0 = 4 > 0 \Rightarrow y = 1$$

If we want to trade positions of $y=0$ and $y=1$ regions we use $-(w_0, w_1, w_2)$.

Rules for signs:



moral: (w_1, w_2) points toward $y=1$ region

$w_0 < 0$ (w_1, w_2) vec crosses decision boundary

$w_0 > 0$ " " doesn't cross " "