EX: An engineer is trying to analyze a complicated digital chip for which a detailed logic diagram is unavailable. The engineer wants to use probabilities to describe the relationships between bit patterns appearing at different points on the chip.

In particular, there is a place where 2 bits feed into a processing block. After unspecified interactions with other signals, the bits eventually influence the values of 2 bits on a bus connected to the output of the processing block.

The engineer has measured the following probabilities of 0, 1, or 2 bits being high at the inputs of the processor (C):

$P(C = 0) = 0.2$ $\quad$ $P(C = 1) = 0.5$ $\quad$ $P(C = 2) = 0.3$

The engineer has measured the following conditional probabilities of 2 bits being high on the bus (B) given 0, 1, or 2 bits being high at the inputs of the processor (C):

$P(B=2 \mid C=0) = 0.1$ $\quad$ $P(B=2 \mid C=1) = 0.2$ $\quad$ $P(B=2 \mid C=2) = 0.7$

Now the engineer wants to measure the bits on the bus and calculate the probabilities of input patterns to the processors. Calculate one such term: $P(C=2 \mid B=2)$.

SOL'N: Use Bayes' theorem to flip the conditional probability from B given C to C given B.

We need only show that the events C = 0, C = 1, and C = 2 are a total partition of the sample space of possible outcomes. Since there are only 2 input bits to the processor, it follows that the only possibilities are that there are 0, 1, or 2 bits high at the inputs. Thus, these events are exhaustive, (i.e., they include all possible outcomes). Note that, since B represents bit lines distinct from those determining C, the statement C = 0 allows B to have all possible values. In other words, we have

$$P(C = 0) \equiv P(C = 0 \text{ and } B = 0, 1, \text{or } 2)$$

Similar statements apply to $P(C = 1)$ and $P(C = 2)$, and we see that the events C = 0, C = 1, and C = 2 cover the entire sample space.

We also have that the events C = 0, C = 1, and C = 2 are mutually exclusive: only one of them can be true for the input bits.

Thus, the events C = 0, C = 1, and C = 2 are exhaustive and mutually exclusive. By definition, then, the events C = 0, C = 1, and C = 2 form a total partition, allowing us to apply the law of total probability to write the following equation for $P(B = 2)$:

$$P(B=2)=\sum_{i=0}^{2}P(B=2,C=i)=\sum_{i=0}^{2}P(B=2|C=i)P(C=i)$$

This is the heart of Bayes' theorem, which gives the following equation:

$$P(C=2|B=2)=\frac{P(B=2,C=2)}{P(B)}=\frac{P(B=2|C=2)P(C=2)}{\sum_{i=0}^{2}P(B=2|C=i)P(C=i)}$$

Plugging in values given in the problem, we have the following calculation:

$$P(C=2|B=2)=\frac{0.7\cdot 0.3}{0.1\cdot 0.2+0.2\cdot 0.5+0.7\cdot 0.3}=\frac{0.21}{0.33}\cong 0.636$$