EX: Are the bounds given by Chebyshev's inequality more accurate when $f(x)$ is a uniform distribution or when $f(x)$ is a gaussian distribution? Justify your answer.

SOL'N: Chebyshev's inequality is more helpful when a distribution has long tails. The probability density for a uniform distribution drops to zero for x more than a certain number of σ's from the mean, μ. For a uniform distribution from 0 to 1, for example, $\sigma^2 = 1/12$, and $\sigma = 1/\sqrt{12}$. The probability density drops to zero for values farther than 1/2 from $\mu = 1/2$. Solving $c\sigma = 1/2$, we find that $c = \sqrt{3}$. Thus, for a uniform distribution, we have

$$P(\mu - c\sigma \le X \le \mu + c\sigma) = 1 \text{ for } c \ge \sqrt{3}.$$

In this case, Chebyshev's inequality only guarantees a probability of

$$P(\mu - c\sigma \le X \le \mu + c\sigma) \ge 1 - \frac{1}{c^2} = \frac{2}{3} \text{ for } c = \sqrt{3}.$$

Thus, Chebyshev's inequality is of little use for a uniform density function.

If we consider a standard gaussian (with $\mu = 0$ and $\sigma = 1$), the probability never drops to zero as we move away from the mean. If, for example, we consider $c = \sqrt{3}$, we can use a table for the area under a standard gaussian to find $P(X \le \mu + c\sigma) = P(X \le \sqrt{3}) \approx P(X \le 1.73) = 0.9582$. We subtract from this $P(X \le \mu - c\sigma) = 0.0418$ to obtain

$$P(\mu - c\sigma \le X \le \mu + c\sigma) = 0.9582 - 0.0418 = 0.9164 \text{ for } c = \sqrt{3}.$$

In this case, Chebyshev's inequality guarantees a probability of

$$P(\mu - c\sigma \le X \le \mu + c\sigma) \ge 1 - \frac{1}{c^2} = \frac{2}{3} = 0.6667 \text{ for } c = \sqrt{3}.$$

This is better than the approximation for the uniform density function, although it still seems rather conservative.