THM: For any probability density function and any real number $k > 0$

$$P\left(\left|X-\mu\right| \geq k\sigma\right) \leq \frac{1}{k^2} \quad \text{or} \quad P\left(\left|X-\mu\right| \leq k\sigma\right) \geq \frac{1}{k^2}$$

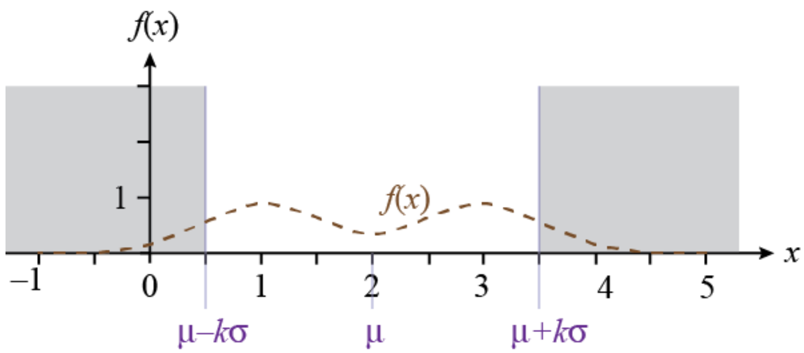
NOTE: This theorem gives an upper bound on how much of the probability density can lie farther than $k\sigma$ from the mean value. Thus, the probability density is constrained in how far its tails can lie from the mean value on a scale measured by standard deviations.

NOTE: This theorem is only useful for values of $k > 1$, since probability is always less than or equal to unity, and the theorem is most useful for larger values of k. For example, all but one-ninth of the probability lies within three standard deviations of the mean, regardless of what the probability density function happens to be.

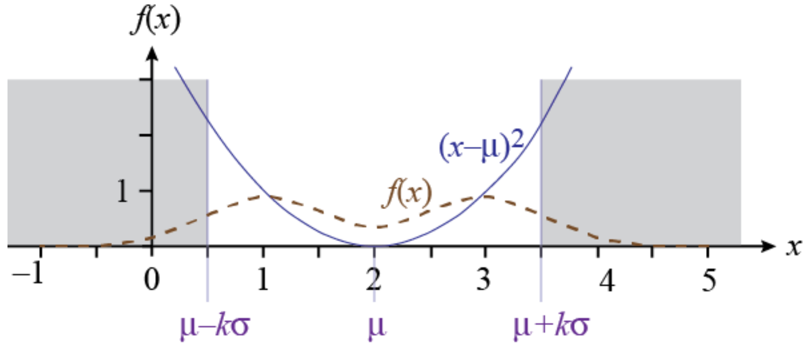
PROOF: We start with the definition of standard deviation:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx\,.$$

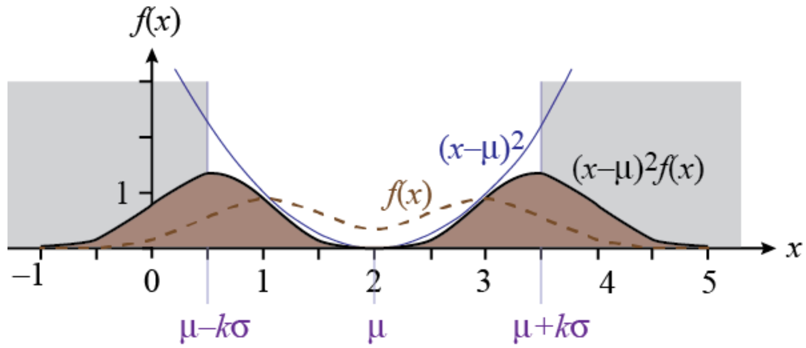
The figure below shows a generic probability density function, $f(x)$.



For the calculation of σ^2, we will multiply $f(x)$ by the quadratic function $(x-\mu)^2$ added to the graph below.



The product $(x-\mu)^2 f(x)$ is shown below, and the area under this curve, (i.e., the integral of $(x-\mu)^2 f(x)$), shown in brown, is the value of σ^2.



We split the integral for σ^2 into regions within $k\sigma$ of the mean (center region) and without $k\sigma$ of the mean (gray regions) giving us the following result.

$$\sigma^2 = \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx\,.$$

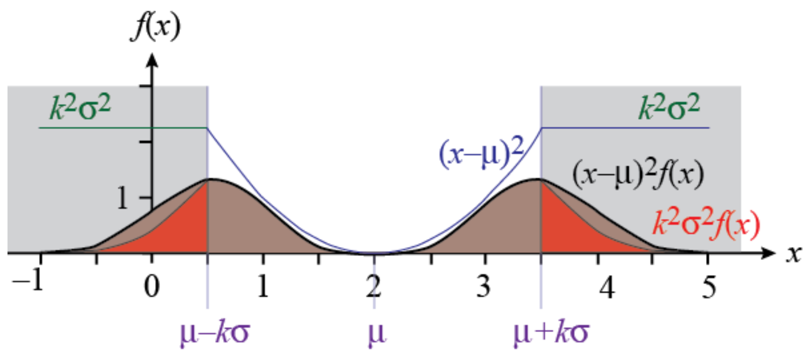
Since the quantities being integrated are all non-negative, if we were to delete the middle integral (i.e., the integral for values within $k\sigma$ of the mean) we would have the following result:

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx\,.$$

In other words, the areas under the side portions are less than the entire area. We obtain an even smaller area on the sides if we replace $(x-\mu)^2$ with a smaller multiplier, namely $k^2\sigma^2$. That is, for the integrals in the above equation we have $(x-\mu)^2 \geq k^2\sigma^2$, so we can write the following inequality:

$$\sigma^2 \geq \int_{-\infty}^{-k\sigma} k^2\sigma^2 f(x)\,dx + \int_{k\sigma}^{\infty} k^2\sigma^2 f(x)\,dx\,.$$

The figure below shows the right-hand side of this equation as red areas that are clearly smaller than the original side areas.



At this point, we factor out the $k^2\sigma^2$ from the integrals to obtain

$$\sigma^2 \geq k^2\sigma^2\left(\int_{-\infty}^{-k\sigma} f(x)\,dx + \int_{k\sigma}^{\infty} f(x)\,dx\right)$$

or, if we divide both sides by σ,

$$1 \geq k^2\left(\int_{-\infty}^{-k\sigma} f(x)\,dx + \int_{k\sigma}^{\infty} f(x)\,dx\right).$$

The value in parentheses is now a probability, and we have

$$1 \geq k^2 P\left(\left|X-\mu\right| \geq k\sigma\right)$$

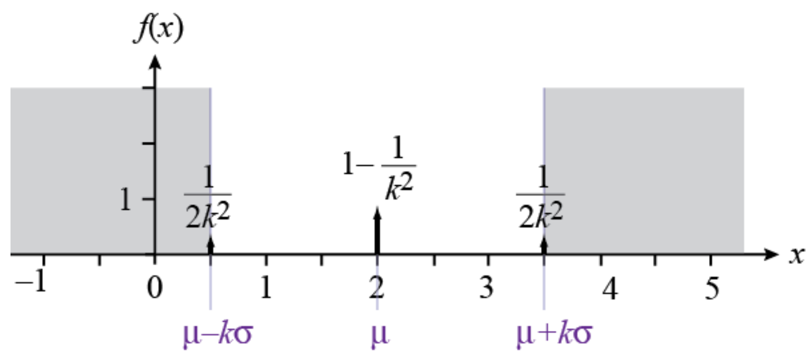
or, if we divide both sides by k^2,

$$\frac{1}{k^2} \geq P\left(\left|X - \mu\right| \geq k\sigma\right).$$

This result is equivalent to the theorem statement, and our proof is finished.

One might wonder the bound is achievable, and the answer for $k > 1$ is yes. The distribution shown below achieves the bound by putting as much of the probability as possible (i.e., $1/2k^2$) at points masses located at distance $k\sigma$ from μ. Thus, we have a discrete distribution:

$$P(X) \text{ or } f(x) = \begin{cases} \frac{1}{2k^2} & x = \mu - k\sigma \\ 1 - \frac{1}{k^2} & x = 0 \\ \frac{1}{2k^2} & x = \mu - k\sigma \\ 0 & \text{otherwise} \end{cases}.$$



We verify that the calculated variance is indeed σ^2:

$$\sigma^2 = \sum_{x_i} (x_i - \mu)^2 P(x_i) = (-k\sigma)^2 \frac{1}{2k^2} + 0 \cdot (1 - \frac{1}{k^2}) + (k\sigma)^2 \frac{1}{2k^2},$$

which simplifies to $\sigma^2 = \sigma^2$, as required.

REF: Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, and Keying Ye, *Probability and Statistics for Engineers and Scientists,* 8th Ed., Upper Saddle River, NJ: Prentice Hall, 2007.