**EX:** An internet router is able to simultaneously route packets on 2 arbitrary input lines to 2 arbitrary output lines so long as no two packets require the same output line. Suppose the router has 10 inputs and 10 outputs. Suppose also that the probability of any input being active at any given time is 0.1, independent of the activity on other inputs. (The probability of being active is the same for all inputs.) The input packets contain information specifying which output line they should be routed to.

a) Find $P(0$ inputs active at a given time).

b) Find $P(0, 1,$ or $2$ inputs active at a given time).

c) Assuming the specified output line for each incoming packet is equally likely to be any one of the ten outputs, independent of the pattern of input activity, find $P(2$ inputs active and collision on output lines occurs at a given time).

**SOL'N:** a) For the sake of brevity, we'll refer to $P(0$ inputs active at a given time) as $P(0$ inputs active). To have $0$ inputs active, we must have the 1st input inactive And the 2nd input inactive And the 3rd input inactive And so forth through the 10th input inactive. For any input, we have

$$P(\text{input inactive}) = 1 - P(\text{input active}) = 1 - 0.1 = 0.9$$

Since the inputs behave independently, we can use the formula for independent events:

$$P(A \text{ and } B) \equiv P(A \cap B) \equiv P(A,B) = P(A)P(B)$$

Thus, the probability of all inputs being inactive is the product of the probabilities for each input being inactive:

$$P(0 \text{ inputs active}) = 0.9^{10} = 0.349$$

b) Since there is no overlap of events with different numbers of active inputs, (i.e., it is impossible to have both $0$ inputs active and $1$ input active at the same time), we can sum probabilities to find $P(0, 1,$ or $2$ inputs active):

$$P(0, 1, \text{ or } 2 \text{ inputs active}) =$$

$$P(0 \text{ inputs active}) + P(1 \text{ input active}) + P(2 \text{ inputs active})$$

We computed the first term in part (a). The calculation of the second and third terms is somewhat more complicated. We first consider

$P$(1 input active).  There is more than one pattern of input activity with 1 input active.  If we pick one particular pattern with 1 input active, such as the first input active and all other inputs inactive, we get the following probability

$$P(1st \text{ input active and inputs 2 thru 10 inactive}) = 0.1 \cdot 0.9^9$$

Any other pattern with exactly one input active will have the same probability:  one input will be active with probability 0.1, while the other nine inputs will be inactive with probability 0.9.  Thus, $P$(1 input active) is equal to the probability of one particular pattern with 1 input active times the number of patterns with exactly one input active.  The number of such patterns of 1 input active out of 10 is given by a combinatorial coefficient:

$$\text{number of patterns with 1 input active} = {}_{10}C_1$$

Thus, we have

$$P(1 \text{ input active}) = 0.1 \cdot 0.9^9 \cdot {}_{10}C_1 = 0.1 \cdot 0.9^9 \cdot 10.$$

Similar reasoning leads us to conclude that

$$P(2 \text{ inputs active}) = 0.1^2 \cdot 0.9^8 \cdot {}_{10}C_2 = 0.1^2 \cdot 0.9^8 \cdot 45.$$

Summing probabilities, we have

$$P(0, 1, \text{ or } 2 \text{ inputs active}) = 0.9^{10} + 0.1 \cdot 0.9^9 \cdot 10 + 0.1^2 \cdot 0.9^8 \cdot 45$$

$$P(0, 1, \text{ or } 2 \text{ inputs active}) \cong 0.3486 + 0.3874 + 0.1937 = 0.930$$

c)  The problem states that the selection of outputs is independent of the pattern of input activity.  Note, however, that one output is selected for each active input.  That is, two inputs active means two outputs are selected.  Thus, we have a product of probabilities:

$$P(2 \text{ inputs active and collision on output}) =$$

$$P(2 \text{ inputs active})P(\text{collision on output [when exactly 2 outputs selected]})$$

We calculated $P$(2 inputs active) in (b).  To find $P$(collision on output) we note that selections of outputs are independent events.  Thus, the probability of a collision when selecting two outputs, is merely the probability of selecting the same output twice in a row, as though we were shaking two ten

sided die and finding the probability of getting doubles.  After we randomly select the first output, the probability of selecting that output again is just one out of ten, or 0.1.  In other words, $P$(collision on output) = 0.1.

Multiplying probabilities gives our final answer:

$$P(2 \text{ inputs active and collision on output}) = 0.1^2 \cdot 0.9^8 \cdot 45 \cdot 0.1 \approx 0.0194$$