EX: Five fair 6-sided dice are rolled. Find the probability that two but no more than two dice show the same number. In other words, the roll yields one or two pairs but not three-of-a-kind.

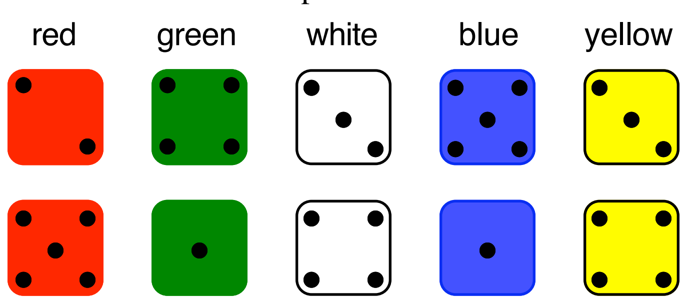
SOL'N: We first consider a direct calculation of the probability. A convenient partition of the sample space distinguishes between one pair, two pairs, and all other outcomes:

$A1 \equiv$ The dice show one and only one pair

$A2 \equiv$ The dice show two pairs (but not four of a kind)

$A3 \equiv$ The dice show something other than one or two pairs

For the sake of calculating probabilities, we may imagine that the dice have distinct colors and are rolled in a particular order, one at a time. This yields the same probabilities as rolling a handful of identical dice all at once. Outcomes with one and two pairs are shown below.



Since all rolls are equally likely (with fair dice) the probability of an event equals the number of outcomes in the event divided by the total number of possible outcomes, which is 6^5.

The number of outcomes in event $A1$ is the product of the following terms:

1. The number of choices for the number showing on the pair = 6.
2. The number of ways of choosing which two dice the pair appear on = ${}_5C_2 = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$.
3. The number of choices for the numbers, which must all be different from each other and the pair, showing on the three dice not in the pair = 5·4·3 = 60.

4. The number of ways of choosing which dice are not in the pair = ${}_3C_3 = 1$. Note that, because the pair are removed from the original set of five dice before we choose which dice are not part of the pair, we are picking from a set of three objects.

The number of outcomes in $A1$ is 6·10·60·1 = 3600.

$$P(A3) = \frac{3600}{6^5} = 0.4630$$

This is about 1 in 2.

The number of outcomes in $A2$ is more difficult to compute owing to the hazard of counting the same pair twice. The *following calculation*, for example, overcounts by a factor of two and *is*, therefore, *wrong*:

Compute the product of the following terms:

1. The number of choices for the number showing on the first pair = 6.
2. The number of ways of choosing which two dice the pair appear on = ${}_5C_2 = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$.
3. The number of choices for the number showing on the second pair = 5, (since it must not be the same number as the first pair).
4. The number of ways of choosing which two dice the second pair appear on = ${}_3C_2 = 3$. Note that, because the first pair are removed from the original set of five dice before we choose the dice for the second pair, we are picking from a set of three objects.
5. The number of choices for the number that is not in either pair = 4.
6. The number of ways of choosing the die is not in withere pair = ${}_1C_1 = 1$. Note that, because the two pair are removed from the original set of five dice before we choose which die is not in either pair, we are picking from a set of one object.

Thus, the *wrong* number of outcomes in $A2$ is 6·10·5·3·4·1 = 3600.

The problem here is that we have counted each of the two pairs twice. In the illustration of two pairs above, containing a pair of fours and a pair of ones, there is nothing to distinguish which pair is the "first" pair. We might have designated either the two fours or the two ones as the first pair. Because of this ambiguity, we mistakenly count both as possible outcomes, even though they are actually the same outcome. Thus, we have overcounted by a factor of two. In general, if we had n pairs to deal with, we would overcount by a factor of ${}_n\mathrm{P}_n = n!$ Note that, because we are considering ways of rearranging the pairs rather than which dice are used, we use the permutation coefficient rather than the combinatoric coefficient.

Thus, the correct number of outcomes in $A2$ is $\dfrac{3600}{2!} = 1800$.

$$P(A2) = \frac{1800}{6^5} = 0.2315$$

This is about 1 in 4.

Summing probabilities, we get our final answer

$$P(\text{one or two pairs in 5 dice}) = \frac{3600 + 1800}{6^5} = \frac{5400}{6^5} = 0.6944$$

This is about 2 in 3.

An alternative approach is to find the probability of *not* getting one or two pairs, P(not one or two pairs), and using 1 – P(not one or two pairs) as our answer.

We can partition the set, B, of outcomes for not getting one or two pairs as follows:

$B1 \equiv$ All five dice show the same number

$B2 \equiv$ Exactly four dice show the same number; (the other die must not be the same number as the four-of-a-kind)

$B3 \equiv$ Exactly three dice show the same number; (the other two may form a pair but must not be the same number as the three-of-a-kind)

$B4 \equiv$ No two dice show the same number

Using the same type of logic as used earlier, we calculate the probabilities for $B1$ through $B4$ by considering the number of ways of choosing which dice show the n-of-a-kind and multiply by 6 for which number is showing. Then we multiply by the number of choices for numbers showing on the remaining dice, being careful to rule out pairs.

The number of outcomes in $B1 = {}_5C_5 \cdot 6 = 1 \cdot 6 = 6$.

The number of outcomes in $B2 = {}_5C_4 \cdot 6 \cdot {}_1C_1 \cdot 5 = 5 \cdot 6 \cdot 1 \cdot 5 = 150$.

The number of outcomes in $B3 = {}_5C_3 \cdot 6 \cdot {}_2C_2 \cdot 5 \cdot 4 = 10 \cdot 6 \cdot 1 \cdot 5 \cdot 5 = 1500$.

The number of outcomes in $B4 = {}_5C_5 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 1 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

The number of outcomes in $B = 6 + 150 + 1500 + 720 = 2376$

$$P(B) = \frac{2376}{6^5} = 0.3056$$

This is about 2 in 3.

$$P(\text{one or two pairs in 5 dice}) = 1 - 0.3056 = 0.6944$$

This is about 2 in 3, as before.