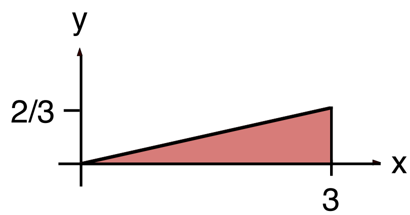
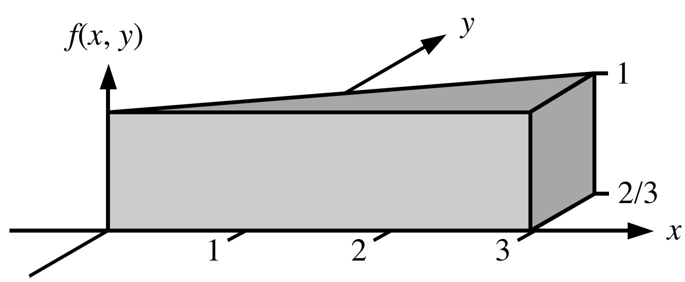
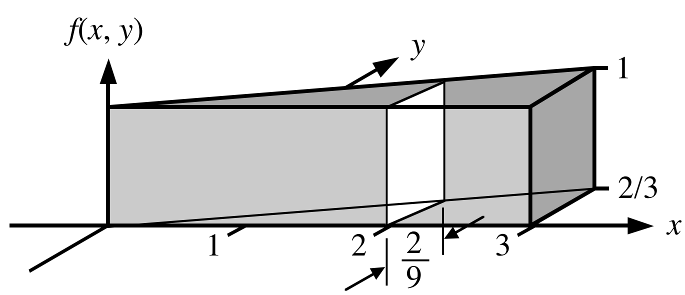
EX: Given joint probability density function $f(x, y) = 1$ on the area of the x,y-plane shown below, find the conditional probability density function $f(y \mid x = 2)$.



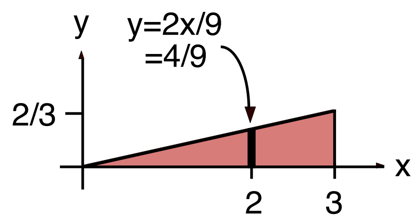
SOL'N: The illustration below shows a 3-dimensional view of $f(x, y)$.



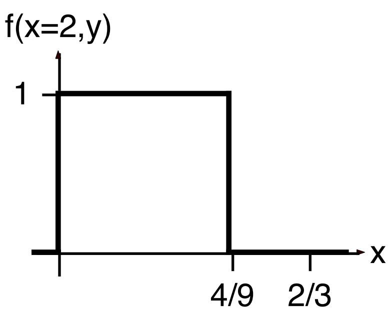
The conditional probability, $f(y \mid x = 2)$, is equal to the cross-section of $f(x, y)$ in the y direction at $x = 2$ scaled vertically to make the area equal to one. The illustration below shows the cross-section at $x = 2$.



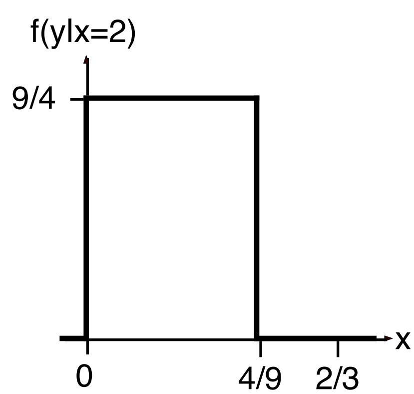
The width of the cross-section is apparent in the following top view of the support (or footprint) of $f(x, y)$ in the xy-plane.



This cross-section is a function of *y* as shown below.



We scale the figure vertically to obtain area equal to one. That is, we multiply by 9/4:



If we take a purely mathematical approach to finding $f(y \mid x = 2)$, we use the definition of conditional probability:

$$f(y \mid x=2) = \begin{cases} \dfrac{f(x=2,y)}{\int_{y=0}^{y=2x/9=4/9} f(x=2,y)\,dy} & 0 \le y \le 4/9 \\ 0 & \text{otherwise} \end{cases}$$

Substituting $f(x = 2, y) = 1$, we complete the calculation:

$$f(y \mid x=2) = \begin{cases} \dfrac{1}{\int_{y=0}^{y=2x/9=4/9} 1\,dy = y\Big|_{y=0}^{y=4/9} = \dfrac{4}{9}} = \dfrac{9}{4} & 0 \le y \le 4/9 \\ 0 & \text{otherwise} \end{cases}$$

The equation for $f(y \mid x = 2)$ also captures the information in the above plot:

$$f(y \mid x=2) = \begin{cases} \dfrac{9}{4} & 0 \le y \le \dfrac{4}{9} \\ 0 & \text{otherwise} \end{cases}$$