**EX:** The following formulas define the behavior of conditional probabilities:

$$P(A\mid B) = \frac{P(A,B)}{P(B)} \equiv \frac{P(A \text{ and } B)}{P(B)} \equiv \frac{P(A\cap B)}{P(B)} \qquad \text{(always true)}$$

$$P(A\mid B) = P(A) \qquad \text{(if } A \text{ and } B \text{ independent)}$$

$$P(A,B) = P(A)P(B) \qquad \text{(if } A \text{ and } B \text{ independent)}$$

For the following formulas, determine whether the formula is always true, could be true in certain cases (be sure to consider all possible ways events $A$ and $B$ might be defined), or is never true. Justify each answer mathematically.

a) $P(A\mid B') = 1 - P(A\mid B)$

b) $P(A,B\mid C) = \dfrac{P(A,B,C)}{P(C)}$

c) $P(A\mid B,C) = P(A,B\mid B,C)$

d) $P(A,B,C) = P(A\mid B,C)P(B\mid C)P(C)$

**SOL'N:** a) We first rearrange the stated formula:

$$P(A\mid B) + P(A\mid B') = 1$$

Using the formula for conditional probability, we substitute for each of the terms on the left side of the equation:

$$\frac{P(A\cap B)}{P(B)} + \frac{P(A\cap B')}{P(B')} = 1$$

The numerators resemble the law of total probability:

$$P(A\cap B) + P(A\cap B') = P(A)$$

We also have $P(B') = 1 - P(B)$. If $P(B) = \frac{1}{2}$, then $P(B') = \frac{1}{2}$ and we have

$$\frac{P(A\cap B)}{\frac{1}{2}} + \frac{P(A\cap B')}{\frac{1}{2}} = 2P(A)$$

which is valid if $P(A) = \frac{1}{2}$. Thus, it is possible for the equation to be valid.

Another case that works is when $A = B$. Then we have the following:

$$P(A \mid B) + P(A \mid B') = P(A \mid A) + P(A \mid A') = 1 + 0 = 1$$

The equation need not always be true, however. Consider the following example of rolling a fair, 6-sided die:

$A \equiv$ The number rolled on the die is 1

$B \equiv$ The number rolled on the die is odd

$B' \equiv$ The number rolled on the die is even

In this case, $P(A \mid B) = \frac{1}{3}$ and $P(A \mid B') = 0$. The sum is not equal to 1, and the equation is invalid. Thus the equation is *sometimes* true.

Exploring further, we might ask if there are restrictions on possible values of $P(A)$ and $P(B)$. We consider the case that $P(B) = \frac{1}{3}$. Then $P(B') = \frac{2}{3}$ and we have the following equation:

$$\frac{P(A \cap B)}{\frac{1}{3}} + \frac{P(A \cap B')}{\frac{2}{3}} = 1$$

This will be satisfied if, for example, $P(A \cap B) = \frac{1}{6}$ and $P(A \cap B') = \frac{1}{3}$. Using the example of rolling a fair, 6-sided die again, we could define $A$ and $B$ as follows:

$A \equiv$ The number rolled on the die is odd

$B \equiv$ The number rolled on the die is 1 or 2

$B' \equiv$ The number rolled on the die is 3, 4, 5, or 6

We conclude that any value of $P(B)$ should work if event $A$ is chosen properly. From the examples given above, it also appears likely that $P(A)$ may have any value if event $B$ is chosen properly. Thus, giving a precise definition of the relationship between $P(A)$ and $P(B)$ that is more informative than the equation in the problem statement is a difficult task.

b) We use the first equation for calculating conditional probability:

$$P(A,B \mid C) = \frac{P((A,B),C)}{P(C)} = \frac{P((A \cap B) \cap C)}{P(C)}$$
$$= \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A,B,C)}{P(C)}$$

We see that $P(A,B \mid C) = \dfrac{P(A,B,C)}{P(C)}$ is always true. We read $(A, B)$ as $A$ and $B$ or as $A \cap B$. We may define this as a new event that is the intersection of two events. It may be helpful to think of a concrete example:

$A \equiv$ It will snow today
$B \equiv$ It will be very cloudy today
$C \equiv$ It is cold today, (as the high temperature today will be < 32° F)

In this case, $(A, B)$ is the event that it will snow and be very cloudy today.

c) We use the first equation for calculating conditional probability applied to the expression on the right side:

$$\begin{aligned} P(A,B \mid B,C) &= \frac{P((A,B),(B,C))}{P(B,C)} = \frac{P((A \cap B) \cap (B \cap C))}{P(B \cap C)} \\ &= \frac{P(A \cap (B \cap B) \cap C)}{P(B \cap C)} = \frac{P(A \cap B \cap C)}{P(B \cap C)} \\ &= \frac{P(A \cap (B \cap C))}{P(B \cap C)} = P(A \mid B,C) \end{aligned}$$

Thus, the equation is always true. The key step in the derivation is realizing that $B \cap B = B$.

Using the concrete example from (b), we can translate $P(A,B \mid B,C)$ into words as the "probability that it will snow today and be very cloudy today given that it is very cloudy and cold today". It makes sense that, since we *know* it is cloudy today, the probability that it will snow *and* be cloudy is the same as the probability that it will snow. An even simpler example of this would be the equation $P(B \mid B) = 1$. If we are given that it is very cloudy today, the probability that it is very cloudy today is equal to 1.

d) This equation is always true, and it follows from repeated application of the first equation for conditional probability rearranged slightly:

$$P(A,B) = P(A \mid B)P(B)$$

Working from right to left in the formula as stated in the problem we have the following:

$$P(B,C) = P(B \mid C)P(C)$$

Proceeding to the next step, we have the final result we seek:

$$P(A \mid B,C)P(B \mid C)P(C) = P(A \mid B,C)P(B,C) = P(A,B,C)$$

Thus, we see that the equation stated in the problem always holds.