Ex:

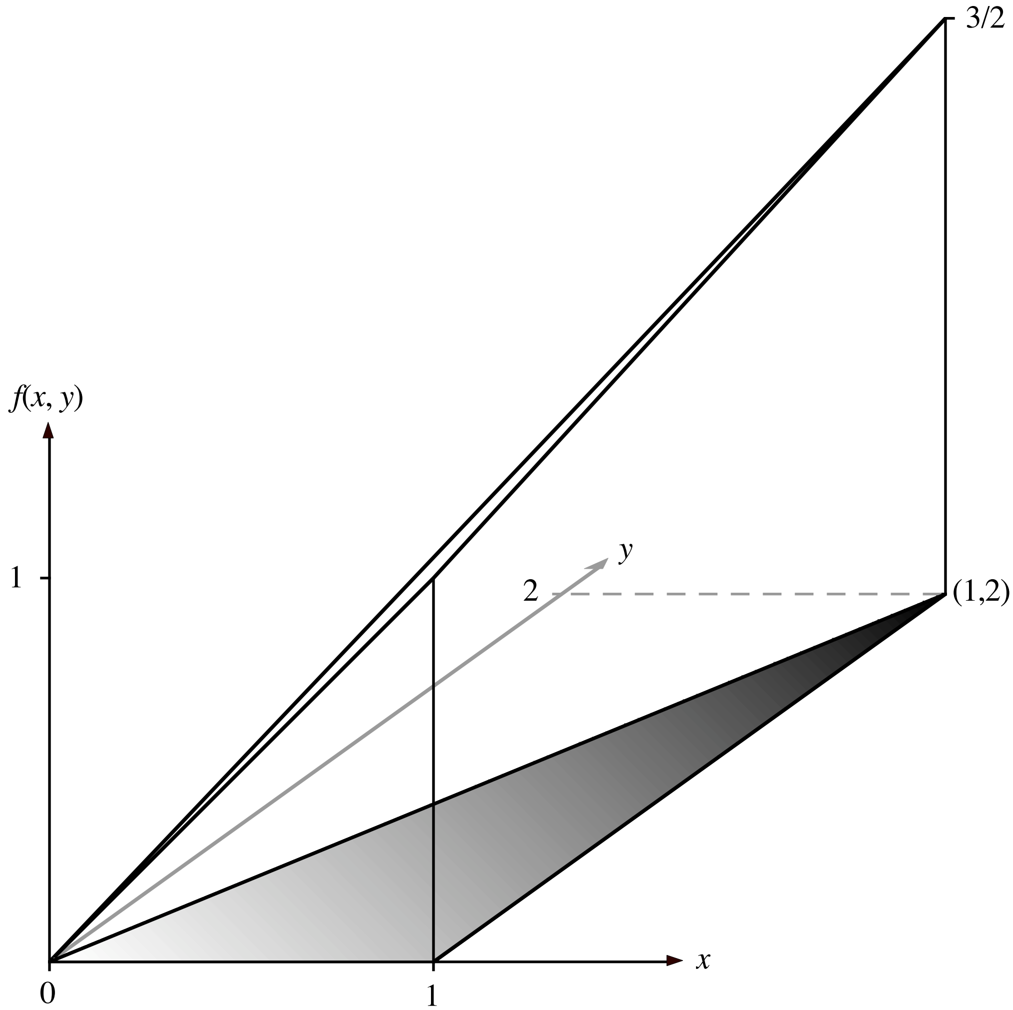

Fig. 1. 3-dimensional view of $f(x, y)$.

Find the covariance, σ_{XY}, of X and Y given the joint probability density function, $f(x, y)$, illustrated above and described by the following equation:

$$f(x) = \begin{cases} \frac{3}{4}(x+y) & 0 \le x \le 1 \text{ and } 0 \le y \le 2x \\ 0 & \text{otherwise} \end{cases}$$

SOL'N: The bottom of the joint probability density function, shown shaded in Fig. 1, is shown below. This is the set of (x, y) values where $f(x, y) \neq 0$, (also known informally as the footprint of $f(x, y)$ and formally as the support of $f(x, y)$). This is what we would see looking down on Fig. 1 from the top.

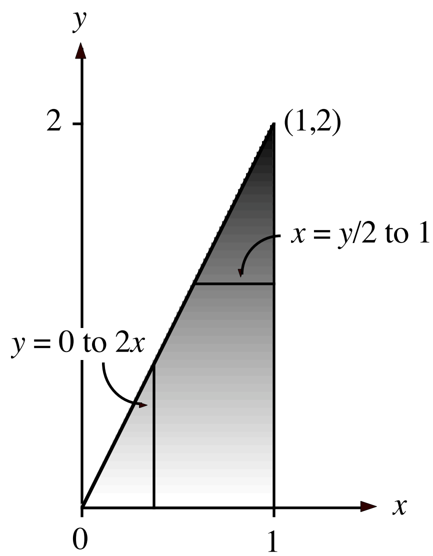

Fig. 2. Support (or footprint) of $f(x, y)$.

We use Fig. 2 to determine the limits of integrals in the calculation of the covariance, σ_{XY}:

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

or, using the definitions for each term,

$$\sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy - \int_{-\infty}^{\infty} xf_X(x)\,dx \cdot \int_{-\infty}^{\infty} yf_Y(y)\,dy.$$

The first term on the right side, $E(XY)$, involves two integrations, the first of which has limits for x that depend on the value of y as shown in Fig. 2:

$$E(XY) = \int_0^2 \int_{y/2}^1 xyf(x,y)\,dx\,dy$$

NOTE: The outer integral, in this case for y, has limits that cover the entire range of possible y values. Also, having y in the limits of the inner integral is acceptable since y looks like a constant when we integrate with respect to x.

NOTE: The limits of the integrals depend only on the *support* (or footprint) of $f(x, y)$. These functions describing the support are distinct from $f(x, y)$. The functions describing the support refer to values in the x,y-plane, whereas $f(x, y)$ describes values in the z direction, or height. Consequently, the functions appearing in the limits of the integrals are typically of a completely different form than $f(x, y)$.

Now we substitute for $f(x, y)$:

$$E(XY) = \int_0^2 \int_{y/2}^1 xy \frac{3}{4}(x+y)dxdy$$

To evaluate the inner integral, we treat y as a constant.

$$E(XY) = \int_0^2 y \frac{3}{4}\left(\frac{x^3}{3} + \frac{x^2}{2}y\right)\Bigg|_{x=y/2}^{x=1} dy$$

Substituting the limits of integration for x and x only, (the y's in the integrand remain unchanged), we obtain the expression for the integral over y:

$$E(XY) = \int_0^2 y \frac{3}{4}\left[\left(\frac{1^3}{3} + \frac{1^2}{2}y\right) - \left(\frac{(y/2)^3}{3} + \frac{(y/2)^2}{2}y\right)\right]dy$$

or

$$E(XY) = \int_0^2 \frac{3}{4}\left(\frac{y}{3} + \frac{y^2}{2} - \frac{y^4}{24} - \frac{y^4}{8}\right) dy$$

or

$$E(XY) = \int_0^2 \frac{3}{4}\left(\frac{y}{3} + \frac{y^2}{2} - \frac{y^4}{6}\right) dy$$

or

$$E(XY) = \frac{3}{4} \cdot \frac{1}{6} \int_0^2 \left(2y + 3y^2 - y^4\right) dy$$

Performing the integration gives the following result:

$$E(XY) = \frac{3}{24}\left(2\frac{y^2}{2} + 3\frac{y^3}{3} - \frac{y^5}{5}\right)\Bigg|_0^2$$

or

$$E(XY) = \frac{1}{8}\left(2\cdot\frac{2^2}{2} + 3\frac{2^3}{3} - \frac{2^5}{5}\right) = \frac{1}{8}\left(4 + 8 - \frac{32}{5}\right) = \frac{1}{8}\left(\frac{20}{5} + \frac{40}{5} - \frac{32}{5}\right)$$

or

$$E(XY) = \frac{7}{10}$$

NOTE: This value for $E(XY)$ is feasible since it lies between the extremes of 0 and 2 that we obtain from the lower-left corner of the support, (i.e., (0, 0)) and the upper-right corner of the support, (i.e., (2,1)). The average value of xy must lie somewhere between these extremes.

Since we may perform the double integration in either order—over x and then over y or over y and then over x—it may be expedient to integrate over y first. This also serves as a consistency check. Here, we perform the integration as a consistency check and to illustrate the method.

If we integrate over y first, our limits of integration are $y = 0$ to $2x$, as shown in Fig. 2.

$$E(XY) = \int_0^1\int_0^{2x} xyf(x,y)dydx$$

or

$$E(XY) = \int_0^1\int_0^{2x} xy\frac{3}{4}(x+y)dydx$$

or

$$E(XY) = \int_0^1 x\frac{3}{4}\left(x\frac{y^2}{2} + \frac{y^3}{3}\right)\Bigg|_{y=0}^{y=2x} dx$$

or

$$E(XY) = \int_0^1 x \frac{3}{4}\left(\frac{x(2x)^2}{2} + \frac{(2x)^3}{3} y\right)dx$$

or

$$E(XY) = \int_0^1 \frac{3}{4}\left(\frac{4x^4}{2} + \frac{8x^4}{3}\right) dx$$

or

$$E(XY) = \int_0^1 \frac{3}{4}\left(\frac{12x^4}{6} + \frac{16x^4}{6}\right) dx$$

or

$$E(XY) = \frac{7}{2}\int_0^1 x^4 \, dx$$

Performing the integration gives the following result:

$$E(XY) = \frac{7}{2}\frac{x^5}{5}\bigg|_0^1$$

or

$$E(XY) = \frac{7}{10}$$

As expected, we get the same answer as before.

Now we turn to the problem of calculating μ_X and μ_Y. We begin by finding the marginal density functions $f_X(x)$ and $f_Y(y)$. These marginal density functions are what we obtain if we ignore the other variable. This is equivalent to lumping together the probability density for all the possible values of the ignored variable. Mathematically, we integrate over the variable we wish to ignore:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \qquad \text{and} \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

Using the information in Fig. 2, we can write the proper limits for the integrals:

$$f_X(x) = \begin{cases} \int_0^{2x} \frac{3}{4}(x+y)dy & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \int_{y/2}^{1} \frac{3}{4}(x+y)dx & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

We treat *y* as a constant when integrating with respect to *x*, and we treat *x* as a constant when integrating with respect to *y*.

$$f_X(x) = \begin{cases} \frac{3}{4}\left(xy + \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=2x} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{3}{4}\left(\frac{x^2}{2} + xy\right)\Bigg|_{x=y/2}^{x=1} & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

The results of the integrals are functions, since they are probability density functions.

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{3}{4}\left[\frac{1}{2} + y - \left(\frac{(y/2)^2}{2} + \frac{y^2}{2}\right)\right] & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

or

$$f_Y(y) = \begin{cases} \frac{3}{4}\left[\frac{1}{2} + y - \frac{5y^2}{8}\right] & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Now we find the mean (or expected) value of X and Y by employing the definition of expected value for a probability density function of a single variable.

$$\mu_X \equiv E(X) = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^1 x \cdot 3x^2 dx = \left.\frac{3x^4}{4}\right|_0^1 = \frac{3}{4}$$

and

$$\mu_Y \equiv E(Y) = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_0^2 y \frac{3}{4}\left[\frac{1}{2} + y - \frac{5y^2}{8}\right]dy$$

or

$$\mu_Y = \left.\frac{3}{4}\left(\frac{1}{2}\frac{y^2}{2} + \frac{y^3}{3} - \frac{5}{8}\frac{y^4}{4}\right)\right|_0^2 = \frac{3}{4}\left(1 + \frac{8}{3} - \frac{5}{2}\right) = \frac{7}{8}$$

Finally, we can compute the covariance, σ_{XY}:

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{7}{10} - \frac{3}{4}\frac{7}{8} = \frac{112 - 105}{160} = \frac{7}{160}$$