EX: Find the covariance, $\sigma_{X\,Y}$, for the following joint probability density function:

$$f(x,y) = \begin{cases} \dfrac{x+y}{3} & 1 \le x \le 2 \text{ and } 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

SOL'N: We start with a calculation of the mean value for X.

$$\mu_X = \int_1^2 x f_X(x)dx = \int_{x=1}^{x=2} x \int_{y=1}^{y=2} \frac{(x+y)}{3} dydx = \int_{x=0}^{x=1} x\left(\frac{x}{3} + \frac{1}{2}\right)dx = \frac{55}{36}$$

Note that the quantity in the parentheses in the last integral is $f_X(x)$.

By symmetry we have $\mu_Y = \mu_X = 55/36$.

Now we compute the covariance. The first step is to find $E(XY)$.

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dydx = \int_{x=1}^{x=2} x \int_{y=1}^{y=2} y \frac{(x+y)}{3} dydx$$

or

$$E(XY) = \int_{x=1}^{x=2} x\left(\frac{1}{2}x + \frac{7}{9}\right)dx = \frac{7}{3}$$

Now subtract the product of the means.

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{7}{3} - \left(\frac{55}{36}\right)^2 = \frac{-1}{1296}$$