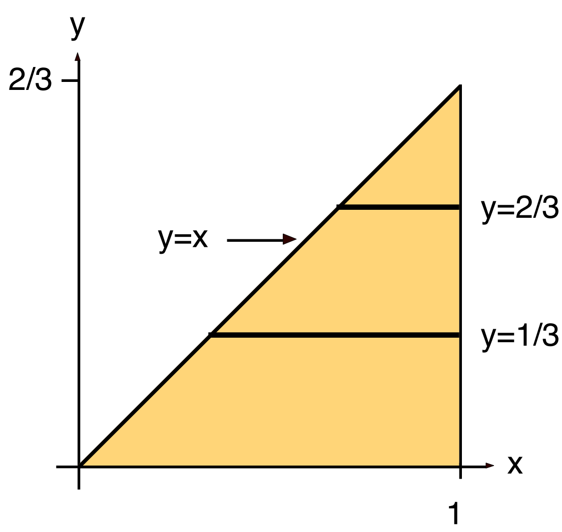
EX: For the following joint probability density function, $f(x, y)$, are X and Y independent? (k is a scaling constant that makes the volume under $f(x, y)$ equal to one.) Explain your answer. If X and Y are independent, find $f_X(x)$ and $f_Y(y)$.

$$f(x,y) = \begin{cases} k(x+y)(x-y) & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \text{ and } x > \text{y} \\ \\ 0 & \text{otherwise} \end{cases}$$

SOL'N: We could try to write $f(x, y)$ as a product of $f_X(x)$ and $f_Y(y)$, but we can demonstrate that X and Y independent by showing that cross sections of $f(x, y)$ have different shapes, even after being scaled to have area = 1. Such scaled cross sections are conditional probabilities. If we get different functions for conditional probabilities at different values of the variable conditioned on, then X and Y are dependent.

One strong clue that X and Y are dependent is that the support (or footprint) of $f(x, y)$ on the xy-plane is wedge-shaped, as shown below. The figure also shows the support for cross sections of $f(x, y)$ at $y = 1/3$ and $y = 2/3$.



We consider $f(x \mid y = \frac{1}{3})$ and $f(x \mid y = \frac{2}{3})$ and show that the former is nonzero for values of x where the latter is zero.

Using the definition of $f(x \mid y = \frac{1}{3})$, we have the following equation:

$$f(x \mid y = \frac{1}{3}) = \frac{f(x, y = \frac{1}{3})}{\int_{x=1/3}^{x=1} f(x, y = \frac{1}{3}) dx}$$

The integral in the denominator yields a value that is some positive real constant that scales $f(x, y = 1/3)$ vertically to ensure that the area under the conditional probability density function equals one. Since we are concerned with where the conditional probability is nonzero, we will focus on the numerator.

Referring back to the figure of the support of $f(x, y)$, above, we have nonzero values for $f(x, y = 1/3)$ for $\frac{1}{3} < x \le 1$:

$$f(x, y = \frac{1}{3}) = k(x + \frac{1}{3})(x - \frac{1}{3}) \text{ for } \frac{1}{3} < x \le 1$$

Following a similar approach for $y = 2/3$, we have nonzero values on a different interval of x values:

$$f(x, y = \frac{2}{3}) = k(x + \frac{2}{3})(x - \frac{2}{3}) \text{ for } \frac{2}{3} < x \le 1$$

In both cases, we observe that $x > y$ guarantees that $f(x, y) > 0$ on the intervals of support for both $f(x, y = 1/3)$ and $f(x, y = 2/3)$.

We also observe that $f(x, y = 1/3)$ has nonzero values on the interval $1/3 < x \le 2/3$, whereas $f(x, y = 2/3)$ has zero values on that interval. It follows that the conditional probabilities are unequal, regardless of vertical scaling that may occur:

$$f(x \mid y = \frac{1}{3}) \ne f(x \mid y = \frac{2}{3})$$

Thus, X and Y are **dependent**.