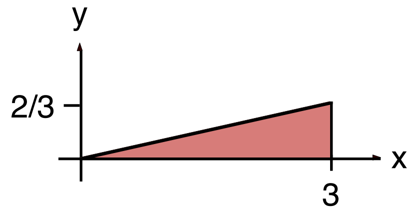
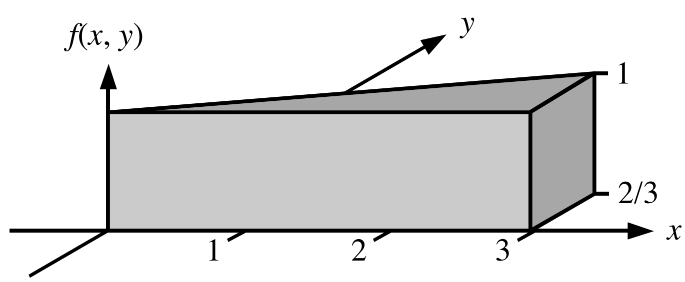
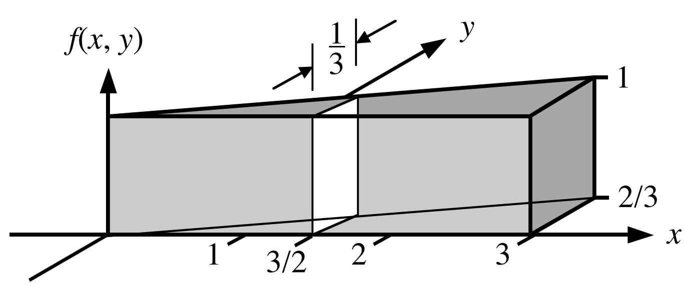
EX: Given joint probability density function $f(x, y) = 1$ on the area of the x,y-plane shown below, find the marginal probability density functions, $f_X(x)$ and $f_Y(y)$.



SOL'N: The illustration below shows a 3-dimensional view of $f(x, y)$.



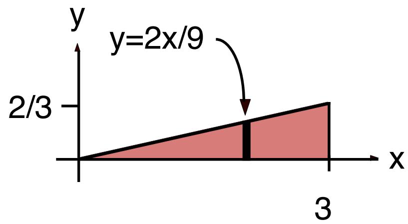
The value of $f_X(x)$ at a given value of x is the area of the cross section of $f(x, y)$ in the y direction. In the illustration below, the value of $f_X(x = 3/2)$ is shown to be equal to $1/3 \cdot 1$ (i.e., width $\cdot$ height) = 1/3.



Since the cross-sectional area has a width that grows linearly as x increases from 0 to 3, we can write down a formula for $f_X(x)$ directly:

$$f_X(x) = \begin{cases} \frac{2}{9}x & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Mathematically, we get the same answer by integrating $f(x, y)$ in the y direction. We must, however, correctly determine the limits of integration. We do so by considering a top view of the support (or footprint) of $f(x, y)$ on the xy-plane:



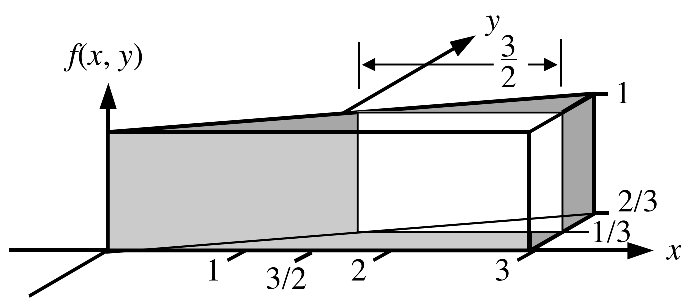
For a given value of x between 0 and 3, y has values between $y = 0$ and $y = \frac{2}{9}x$. Thus, the upper limit of the integral for $f_X(x)$ depends on x:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{y=0}^{y=2x/9} f(x,y)dy = \int_{y=0}^{y=2x/9} 1dy = y\Big|_{y=0}^{y=2x/9}$$

Completing the calculation, we get our answer, (which is the same as before):

$$f_X(x) = \begin{cases} \frac{2}{9}x & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

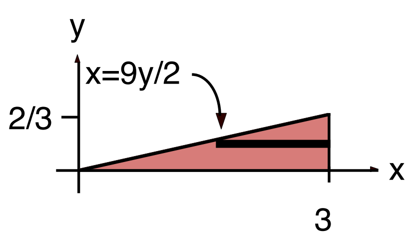
Similar arguments apply for the calculation of $f_Y(y)$. The graphical approach relies on calculation of areas of cross sections in the x direction. In contrast to cross sections in the y direction, the area of the cross sections in the x direction decrease in area as y increases. The diagram below shows that the cross section for $y = 1/3$ has area equal to $3/2 \cdot 1$ (i.e., width $\cdot$ height) = 3/2.



Since the cross-sectional area has a width that decreases linearly as y increases from 0 to 2/3, we can write down a formula for $f_Y(y)$ directly:

$$f_Y(y) = \begin{cases} \frac{9}{2}\left(\frac{2}{3} - y\right) & 0 \le y \le \frac{2}{3} \\ 0 & \text{otherwise} \end{cases}$$

Mathematically, we get the same answer by integrating $f(x, y)$ in the x direction. As before, we must correctly determine the limits of integration. From the top view of the support (or footprint) of $f(x, y)$ on the xy-plane we see that, for a given value of y between 0 and 2/3, x has values between $x = \frac{9}{2}y$ and $x = 3$.



This time, the lower limit of the integral for $f_Y(y)$ depends on y:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{x=9y/2}^{x=3} f(x,y)dx = \int_{x=9y/2}^{x=3} 1dx = x\Big|_{x=9y/2}^{x=3}$$

Completing the calculation, we get our answer, (which is the same as before):

$$f_Y(y) = \begin{cases} 3 - \frac{9}{2}y & 0 \le y \le \frac{2}{3} \\ 0 & \text{otherwise} \end{cases}$$