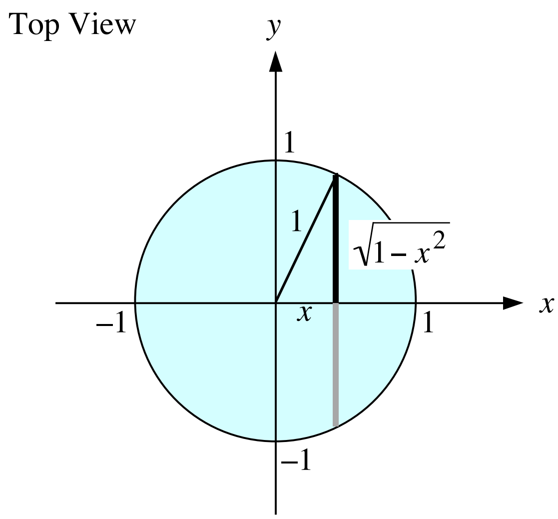
EX: A joint probability density function is defined as follows:

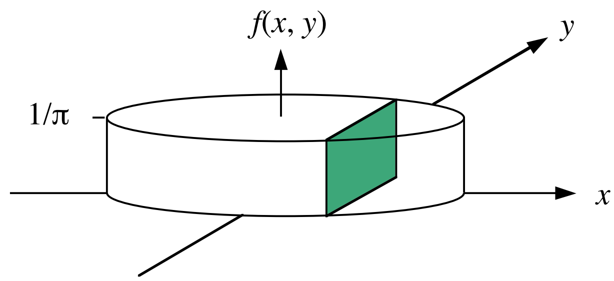
$$f(x,y) = \begin{cases} \dfrac{1}{\pi} & x^2 + y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal probability density functions, $f_X(x)$ and $f_Y(y)$.

SOL'N: The region, $x^2 + y^2 \le 1$, on which $f(x, y) \ne 0$ is called the support of $f(x, y)$. It is a circle of radius one, centered on the origin, as shown below. The diagram also shows the calculation of the vertical segment over which $f(x, y) \ne 0$ as a function of position x.



The illustration, below, shows the 3-dimensional shape of $f(x, y)$ with a height of $k = 1/\pi$. The figure also shows a cross section in the y direction at one value of x.



The value of $f_X(x)$ at one value of x is equal to the area of the cross-section of $f(x, y)$ in the y direction at that value of x. We use the diagram of the support (or footprint) of $f(x, y)$ to determine the limits of integration in the following calculation that determines the area of the cross-section.

$$f_X(x) = \begin{cases} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} f(x,y)dy & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that the restriction on x is the entire width of the figure, since the cross-sections in the y direction are nonzero over this width.

$f(x, y) = 1/\pi$ over the interval of integration. Thus, the value of the integral is just the length of the interval of integration times $1/\pi$:

$$f_X(x) = \begin{cases} \frac{2\sqrt{1-x^2}}{\pi} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

By symmetry, we obtain a similar result for $f_Y(y)$:

$$f_Y(y) = \begin{cases} \frac{2\sqrt{1-y^2}}{\pi} & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$