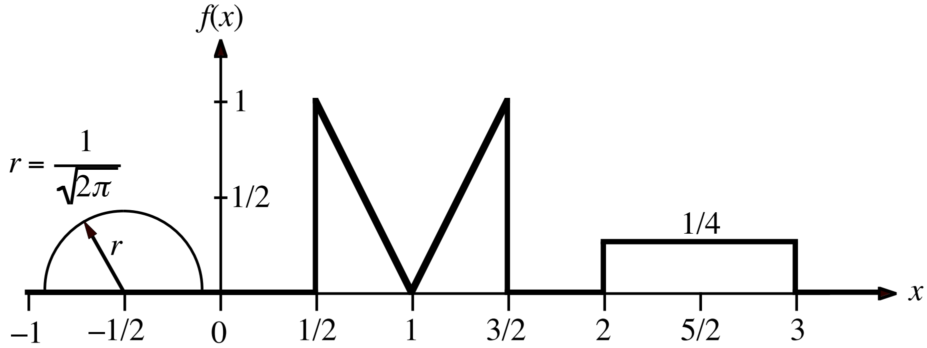
EX: A probability density function, $f(X)$, is shown below. Use the center of mass method to find $E(X)$, the expected value of X.



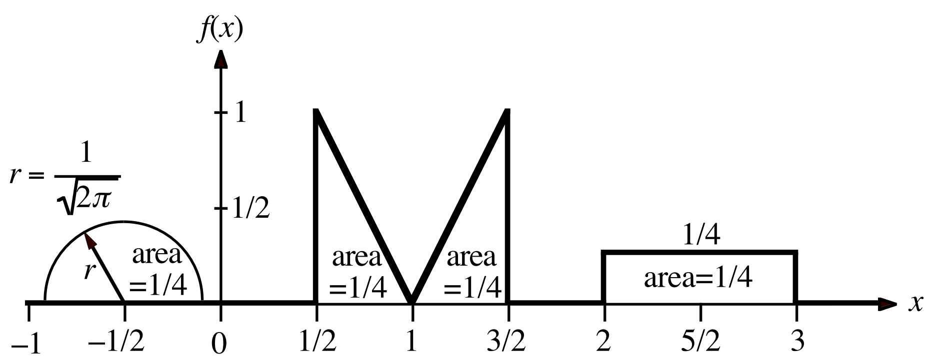
SOL'N: When parts of $f(X)$ are horizontally symmetrical, we can replace them with a point mass located at their center of mass. The value of the point mass is the area of that portion of $f(X)$.

Mathematically, the point mass is represented by a delta (or impulse) function:

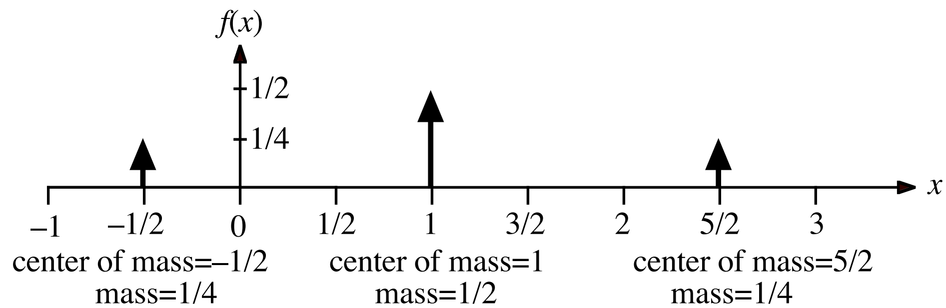
$$m\delta(x-c)$$

where $m \equiv$ mass and $c \equiv$ location of center of mass

For the $f(X)$ given in this problem, the half circle has an area of 1/4 and is centered at –1/2. The "M" has an area of 1/4 + 1/4 = 1/2 centered at 1, and the rectangle has an area of 1/4 centered at 5/2.



These areas are equivalent to point masses as shown below:



Mathematically, the new $f(x)$ is a summation of delta functions:

$$f(x) = \frac{1}{4}\delta(x - -\frac{1}{2}) + \frac{1}{2}\delta(x - 1) + \frac{1}{4}\delta(x - \frac{5}{2})$$

Computing the expected value of this new $f(x)$ we have the following formal steps, (the first few steps of which may be bypassed, as explained below):

$$E(X) = \int_{x=-\infty}^{x=\infty} xf(x)dx = \int_{x=-\infty}^{x=\infty} x\left[\frac{1}{4}\delta(x - -\frac{1}{2}) + \frac{1}{2}\delta(x - 1) + \frac{1}{4}\delta(x - \frac{5}{2})\right]dx$$

We apply the following identity several times:

$$\int_{x=-\infty}^{x=\infty} x\delta(x - a)dx = a$$

This yields the following expression that is the sum of center points times centers of mass, (an expression which may be written down directly without going through the preceding steps):

$$E(X) = -\frac{1}{2} \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + \frac{5}{2} \cdot \frac{1}{4} = 1$$

NOTE: The center-of-mass method may be applied to any shapes, but it is simplest in the case where shapes are horizontally symmetric.