EX: Given X~u(0, 1), (i.e., X is uniformly distributed from 0 to 1), find the probability density function, $f_Y(y)$, for Y where

$$Y = 1 - e^{-X}.$$

SOL'N: The transformation from X to Y is a strictly increasing function, $g(X)$:

$$Y = g(X) = 1 - e^{-X}$$

Thus, we may use an identity for nonlinear transformation of random variables:

$$f_Y(y) = f_X(x = g^{-1}(y))\frac{dg^{-1}(y)}{dy}$$

The inverse function, $g^{-1}(y)$, is found by solving for X in terms of Y:

$$X = g^{-1}(Y) = -\ln(1 - Y)$$

Making the substitution for $g^{-1}(y)$, we have the following expression for $f_Y(y)$:

$$f_Y(y) = f_X(x = -\ln(1 - y))\frac{d(-\ln(1 - y))}{dy}$$

To simplify the first term on the right side, we start with the definition of the probability density of X:

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Wherever x appears in the definition of $f_X(x)$, we substitute $g^{-1}(y)$:

$$f_X(g^{-1}(y)) = \begin{cases} 1 & 0 \le g^{-1}(y) \le 1 \\ 0 & \text{otherwise} \end{cases}$$

or

$$f_X(-\ln(1 - y)) = \begin{cases} 1 & 0 \le -\ln(1 - y) \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Now we rewrite the inequality involving $-\ln(1 - y)$ in terms of y:

$$f_X(-\ln(1 - y)) = \begin{cases} 1 & g(0) \le g(-\ln(1 - y)) \le g(1) \\ 0 & \text{otherwise} \end{cases}$$

or

$$f_X(-\ln(1-y)) = \begin{cases} 1 & 1-e^{-0} \le y \le 1-e^{-1} \\ 0 & \text{otherwise} \end{cases}$$

or

$$f_X(-\ln(1-y)) = \begin{cases} 1 & 0 \le y \le 1-e^{-1} \\ 0 & \text{otherwise} \end{cases}$$

NOTE: If $f_X(x)$ is more complicated than the simple uniform density function considered here and has values that are functions of x, then we would also replace those values of x with $g^{-1}(y)$, too.

Consider the following example with the same $g(x)$ as in the present problem:

$$f_X(x) = \begin{cases} 1-x & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

We would substitute $x = g^{-1}(y)$ for every x:

$$f_X(-\ln(1-y)) = \begin{cases} 1 - -\ln(1-y) & 0 \le -\ln(1-y) \le 2 \\ 0 & \text{otherwise} \end{cases}$$

or

$$f_X(x) = \begin{cases} 1+\ln(1-y) & 0 \le y \le 1-e^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Returning to the problem at hand, we now consider the second term of the expression for $f_Y(y)$:

$$f_Y(y) = f_X(x = -\ln(1-y))\frac{d(-\ln(1-y))}{dy}$$

Taking the derivative yields the expression for the second term:

$$\frac{d(-\ln(1-y))}{dy} = -\frac{1}{1-y}(-1) = \frac{1}{1-y}$$

This term will multiply the first term:

$$f_Y(y) = f_X(-\ln(1-y))\frac{1}{1-y} = \begin{cases} 1 & 0 \le y \le 1-e^{-1} \\ 0 & \text{otherwise} \end{cases} \cdot \frac{1}{1-y}$$

or

$$f_Y(y) = \begin{cases} 1 \cdot \dfrac{1}{1-y} & 0 \le y \le 1 - e^{-1} \\ 0 \cdot \dfrac{1}{1-y} & \text{otherwise} \end{cases}$$

or

$$f_Y(y) = \begin{cases} \dfrac{1}{1-y} & 0 \le y \le 1 - e^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Note that, although $g(\,)$ involved an exponential and $g^{-1}(\,)$ involved a log function, the expression for $f_Y(y)$ contains neither of these.