EX: An engineer is analyzing a diode circuit in which there is a very small voltage across the diode. The current in the diode is given by the following equation:

$$i_D = I_0(e^{v/V_T} - 1)$$

The voltage is small enough that a quadratic approximation (obtained from a Taylor series expansion) for the exponential is sufficiently accurate:

$$i_D \cong I_0\left[\frac{v}{V_T} + \frac{1}{2}\left(\frac{v}{V_T}\right)^2\right]$$

In the next stage of the circuit, the first order term is removed by a current summation, (or subtraction), but a quadratic noise current remains:

$$i_N \cong I_0 \frac{1}{2}\left(\frac{v}{V_T}\right)^2$$

With these preliminaries given, your task is to use the last equation and find the probability density function for i_N, assuming $X \equiv v/V_T$ has a random value uniformly distributed between 1 and 2, (i.e., $X \sim u[1,2]$). To make matters simpler, you need only show that the probability density function for $i_N = X^2 \cdot I_0/2$ is given by the following expression:

$$f(i_N) = \begin{cases} \dfrac{1}{\sqrt{2I_0 i_N}} & \dfrac{I_0}{2} \le i_N \le 2I_0 \\ 0 & \text{otherwise} \end{cases}$$

Hint: Define $Y \equiv i_N \equiv I_0/2 \cdot X^2$. Then take the derivative (d/dy) of the cumulative distribution function for y, $F(y)$, defined in terms of X. In other words, fill in the ?'s and $f(x)$) in the following equation:

$$F(Y) = P(Y \le y) = P(\frac{I_0}{2}X^2 \le y) = P(X \le \sqrt{2y/I_0}) = F_X(\sqrt{2y/I_0}) = \int_?^? f(x)dx$$

SOL'N: First, we observe that the value of cumulative distribution at x, (i.e., $F_X(x)$), for a uniform distribution on (1, 2) is the area to the left of x and will grow linearly from 0 to 1 as x goes from 1 to 2.

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \int_1^x 1 \cdot dx & 1 < x < 2 \\ 1 & x > 2 \end{cases}$$

Thus, we have the following expression for $F_X(x)$:

$$F_X(x) = \begin{cases} 0 & x < 1 \\ x - 1 & 1 < x < 2 \\ 1 & x > 2 \end{cases}$$

Substituting $\sqrt{2y/I_0}$ for x, we have the expression for $F_Y(y)$:

$$F_Y(y) = \begin{cases} 0 & \sqrt{2y/I_0} < 1 \\ \sqrt{2y/I_0} - 1 & 1 < \sqrt{2y/I_0} < 2 \\ 1 & \sqrt{2y/I_0} > 2 \end{cases}$$

We translate the inequalities into expressions for y:

$$F_Y(y) = \begin{cases} 0 & y < I_0/2 \\ \sqrt{2y/I_0} - 1 & I_0/2 < y < 2I_0 \\ 1 & y > 2I_0 \end{cases}$$

Taking the derivative gives $f_Y(y)$:

$$f_Y(y) = \begin{cases} 0 & y < I_0/2 \\ \dfrac{1}{\sqrt{2y/I_0}}\sqrt{2/I_0} & I_0/2 < y < 2I_0 \\ 1 & y > 2I_0 \end{cases}$$

or

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{y}} & \dfrac{I_0}{2} \le y \le 2I_0 \\ 0 & \text{otherwise} \end{cases}$$

This is the result given in the problem statement when we substitute i_N for y.