**EX:** In a communication system, single bits are communicated by short snippets of waveforms of different shapes for 0's and 1's. At the receiving end, the waveform for one bit is sampled 8 times. Noise in the form of gaussian (or normally) distributed values is effectively added to each sample. The noise samples, $X_i$, are independent and identically distributed with mean value $\mu_{X_i} = 0$ V and variance $\sigma_{X_i}^2 = (1.5\text{ V})^2$:

$$X_i \sim n(x_i; 0, 1.5) \quad \text{or} \quad X_i \sim N(0, 2.25)$$

Whether a 0 or 1 is received correctly depends on the total power in the eight noise samples for each bit. This noise power, $W$, is calculated as the sum of the squares of the values of the eight $X_i$'s.

Find the probability density function (**pdf**), $f_W(w)$, of $W$.

**SOL'N:** The sum, $X$, of squares of $\nu$ random variables, $X_i$, with identical, independent, standard gaussian (or standard normal) distributions yields a chi-squared distribution of degree $\nu$, [1]:

$$f_X(x) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu-2)/2} e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

with mean value $\mu = \nu$ and variance $\sigma^2 = 2\nu$.

In this problem, the $X_i$ are independent gaussian random variables but with variances equal to 2.25 instead of 1. Thus, the challenge is to rewrite the sum of squares of the $X_i$ in terms of a sum of squares of standard gaussian (or standard normal) random variables, $Y_i$.

$$W = \sum_{i=1}^{\nu} X_i^2$$

Since the $X_i$ have zero mean, they may be written as scaled standard gaussians. In other words, they are linearly transformed standard gaussians:

$$X_i = aY_i$$

where $a = \sigma_{X_i} = 1.5$.

**NOTE:** A linearly transformed gaussian is a gaussian. If $X = aY + b$, for example, we have the following probability density for $X$:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-(x-\mu_X)^2/2\sigma_X^2}$$

where

$$\mu_X = a\mu_Y + b \quad \text{and} \quad \sigma_X^2 = a^2\sigma_Y^2$$

If we express $W$ in terms of $Y_i$'s, we find that it is a linear transform of a chi-squared random variable:

$$W = \sum_{i=1}^{\nu}(aY_i)^2 = a^2\sum_{i=1}^{\nu}Y_i^2 = a^2X$$

where $X$ is chi-squared of degree $\nu$.

Now we can apply the tool for the pdf of a linear transform of a random variable:

If $W = aX + b$, $(a \neq 0)$, then $f_W(w) = \frac{1}{|a|} f_X\left(x = \frac{w-b}{a}\right)$.

Also, the mean and variance transform as follows:

$$\mu_W = a\mu_X + b, \quad \sigma_W^2 = a^2\sigma_X^2$$

Using this tool, our pdf for $W$ is the following:

$$f_W(w) = \begin{cases} \frac{1}{a}\frac{1}{2^{\nu/2}\Gamma(\nu/2)}\left(\frac{w}{a}\right)^{(\nu-2)/2} e^{-\left(\frac{w}{a}\right)/2} & \frac{w}{a} > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $a = 1.5$.

**REF:** [1] Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, and Keying Ye, *Probability and Statistics for Engineers and Scientists,* 8th Ed., Upper Saddle River, NJ: Prentice Hall, 2007.