May 1990 Real Analysis — Convergence — In Measure

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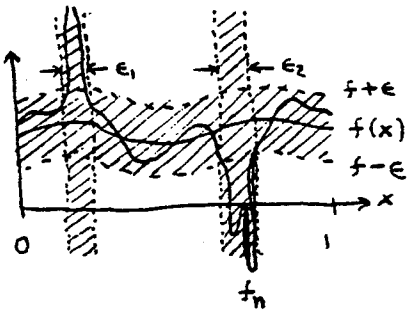
$\langle f_n \rangle \equiv$ sequence of functions on domain D

$\langle f_n \rangle$ converges to f in measure $\equiv$

Given $\epsilon > 0$, there is an N such that

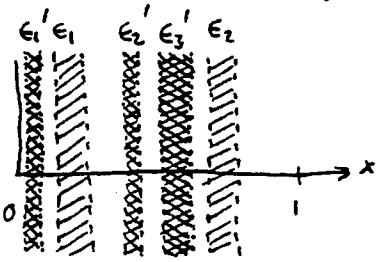
$$m\{x: |f(x) - f_n(x)| \geq \epsilon\} < \epsilon \qquad \text{for all } n \geq N.$$

picture:



Every f_n for $n \geq N$ must be within ϵ of f except on intervals whose total length is at most ϵ, (e.g. $\epsilon_1 + \epsilon_2 \leq \epsilon$ in picture).

Note: the f_n's can miss f by more than ϵ on different intervals for different n's. Thus, the intervals ϵ_1, ϵ_2 in the picture need not be the same for each f_n. A different f_n might miss f by more than ϵ on, say, intervals $\epsilon_1', \epsilon_2', \epsilon_3'$ not even overlapping ϵ_1, ϵ_2 in the picture:



Where f_n misses f by more than ϵ, this miss may be arbitrarily large.

ex: $f_n(x) = \begin{cases} \frac{1}{n} + \delta(x - \frac{1}{2}) & x \in (\frac{1}{n}, 1) \\ 1 & x \in [0, \frac{1}{n}] \end{cases}$ (δ func misses f by ∞ at single point of measa) $\rightarrow f(x) = 0$ in measure

f_n misses $f=0$ by 1 in narrowing plateau region $\rightarrow$