May 1990 Real Analysis — Convergence— In the mean of order p

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$\langle f_n \rangle \equiv$ sequence of functions in L^p on domain D

$\langle f_n \rangle$ converges to f <u>in the mean of order p</u> $\equiv$ $\qquad f \in L^p$

There is an N such that for a given $\epsilon > 0$

$\| f - f_n \|_p < \epsilon$ for all $n \geq N$. $\qquad 1 \leq p < \infty$

$\|g\|_p$ is the L^p norm: $\left\{ \int_D |g|^p \right\}^{1/p}$, $\qquad 1 \leq p < \infty$

$\|g\|_\infty$ = smallest M such that $m\{g > M\} = 0$. Basically the max value of g if points on set of measure 0 excluded

Practical implications of convergence in mean of order p:

If $p = 1$, then the area under $|f - f_n| \to 0$.

If $p = 2$, then the mean squared error of $f - f_n \to 0$.

f may not be bounded, but it cannot be $\pm\infty$ outside a set of measure zero. Otherwise, $\|f\|_p$ would be ∞, and then $f \notin L^p$. By definition, however, $f \in L^p$ for convergence in the mean of order p.

ex: $f \in L^1$ but f not bounded when $f(x) = \frac{1}{\sqrt{x}}$ on $[0,1]$

$$\left\{ \int_0^1 |f(x)| \right\}^{1/1} = \int_0^1 \frac{1}{\sqrt{x}}\, dx \quad (\text{positive root } \sqrt{x}) \qquad \int x^{-1/2} = 2x^{1/2}$$

$$= 2\sqrt{x}\,\Big|_0^1 = 2 \text{ is finite}$$

$\therefore f \in L^1$ but $\frac{1}{\sqrt{x}} \to \infty$ as $x \to 0$ so f not bounded

f unbounded only at $x = 0$, a single point, a set of measure zero.

ref: Royden <u>Real Analysis</u>