May 1990 Real Analysis — Convergence Theorems — Lusin's Theorem

Neil E Cotter

thm (Luzin): Let f be measurable on compact domain D. Assume f finite except on a set of measure zero. (For all thms on this pag

Then there exists a continuous function g such that $f = g$ except on a set of measure $< \epsilon$ where ϵ may be as small as desired.

Alternative notation: $\underset{\text{for all}}{\forall} \epsilon > 0 \ \underset{\text{there exists}}{\exists} g$ cont. $m\{f \neq g\} < \epsilon$

corollary: There exists sequence $\langle f_n \rangle$ of continuous functions such that $\langle f_n \rangle \rightarrow f$ <u>in measure</u>.

thm (in measure $\Rightarrow$ a.e): Let $\langle f_n \rangle$ be a sequence of measurable functions such that $\langle f_n \rangle \rightarrow f$ <u>in measure</u>. Then there is a subsequence $\langle f_{n_k} \rangle$ of $\langle f_n \rangle$ such that $\langle f_{n_k} \rangle \rightarrow f$ <u>a.e.</u>

corollary: If $\langle f_n \rangle$ is a sequence of continuous functions such that $\langle f_n \rangle \rightarrow f$ <u>in measure</u>, then there exists a sequence $\langle f_{n_k} \rangle$ of continuous functions such that $\langle f_{n_k} \rangle \rightarrow f$ <u>a.e.</u>

thm (a.e. $\Rightarrow$ uniform [almost]): Let $\langle f_n \rangle \rightarrow f$ a.e. with f_n measurable. Then for any $\epsilon > 0$ and $\delta > 0$ there exists a set d of measure $m\{d\} < \delta$ such that

$$|f_n(x) - f| < \epsilon \quad \text{for } \underline{\text{all}} \ x \notin d \quad \text{and all } n \geq N.$$

corollary: If $\langle f_{n_k} \rangle \rightarrow f$ <u>a.e.</u> for f_{n_k} continuous then we can find a set d, of arbitrarily small measure, such that $\langle f_{n_k} \rangle \rightarrow f$ <u>uniformly on $D \sim d$</u> (domain excluding d).

ref: H.L. Royden <u>Real Analysis</u> Macmillan 1968 ISBN 0-02-404150-5