May 1990
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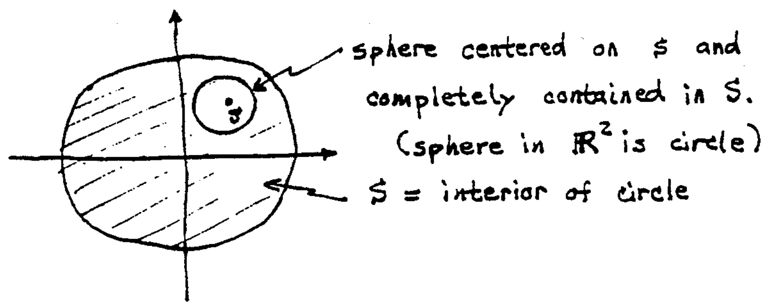
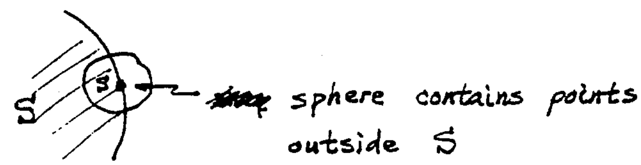
Real Analysis – Sets – Open, Closed, and Compact

S is an open set $\equiv$ For any point $s \in S$ we can find a sphere centered on s and completely contained in S.

ex: Unit circle minus the boundary



Note: If boundary is included then set is not open. Any sphere centered on boundary point will contain points outside unit circle $= S$.



ex: Open interval $S = (0,1)$ is open set in $\mathbb{R}^1$, (0, 1 not in S). Half-open interval $S = (0,1]$ is not open since it includes $x = 1$.

S is a closed set $\equiv$ If x is any point in $\mathbb{R}^n$, then if every sphere around x contains a point in S then x must be in S.

ex: $S = (0,1)$ is <u>not</u> closed since every interval (1-dim sphere) centered at 1 contains a point of S <u>but</u> $1 \notin S$. We call 1 a point of closure for S. Also, 0 is a point of closure for S.

$S \cup$ closure pts of $S \equiv \overline{S}$ is a closed set.

Thus, the closed interval $[0,1]$ is a closed set in $\mathbb{R}^1$.

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Real Analysis - Open, Closed, and Compact ~~Sets~~ Sets — (cont.)

Algebra of sets: $\emptyset$ empty set is open and closed

$\mathbb{R}^n$ whole space " " " "

$$\text{open} \cap \text{open} = \text{open}$$
$$\text{open} \cup \text{open} = \text{open}$$
$$\bigcup_{k=1}^{\infty} \text{open}_k = \text{open}$$

$\cap \equiv$ intersect

$\cup \equiv$ union

$\bigcup_{k=1}^{\infty} \equiv$ countable union of set

$$\text{uncountable } \bigcup_k \text{open}_k = \text{open}$$

$$\text{closed} \cup \text{closed} = \text{closed}$$
$$\text{closed} \cap \text{closed} = \text{closed}$$
$$\bigcap_k \text{closed}_k = \text{closed}$$

countable or uncountable ~~union~~ intersection of set

complement of open = closed

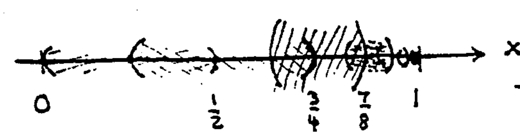
" " closed = open

S is Compact set $\equiv$ Every open covering of S has a finite subcovering of S.

An open covering of S is a collection of open sets ~~that~~ whose union contains S.

ex: $(0,1)$ is covered by $(0, \frac{1}{2}) \cup (\frac{1}{4}, \frac{3}{4}) \cup (\frac{5}{8}, \frac{7}{8}) \cup \ldots$

$$= \bigcup_{k=1}^{\infty} \left(1 - \frac{1}{2^k} - \frac{1}{2^{k+1}}, 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}\right) \cup (0,$$



The union of the intervals covers $(0,1)$

$(0,1)$ is <u>not</u> compact because we cannot find a <u>finite</u> set (of these intervals in the covering) whose union contains all of $(0,1)$.

Sets —

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ex: Open sets are <u>never</u> compact.

ex: Every closed and bounded set in $\mathbb{R}^n$ is compact.

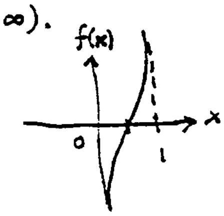
ex: Every compact set of $\mathbb{R}^n$ is closed and bounded.
so $(-\infty, \infty)$ is not comp

ex: $[0, 1]^n$ is compact

ex: Any rectangular region (including boundary) is compact

Intuitively, we need compactness to insure that the domain, D, of functions is not isomorphic to an infinitely long interval. For example, $(0,1)$ can be mapped one-to-one (one pt maps to one point) and onto (every pt is the image of some point under the mapping) the interval $(-\infty, \infty)$.

ex: $f(x) = \tan\left[(x - 1/2)\pi\right]$



range of $f(x)$ is $(-\infty, +\infty)$
domain of $f(x)$ is $(0, 1)$

Note that if we use $[0, 1]$ including endpts then the range of $f(x)$ includes values that are actually infinite: range $= [-\infty, \infty]$. Then, strictly speaking, $f(x)$ is not continuous on $[0,1]$ because it takes on values $\pm\infty$. The same $f(x)$ is continuous on $(0,1)$, however. But $f(x)$ is still a bad function because it is unbounded.

By requiring functions to be continuous on compact domains we eliminate unbounded functions. Note that this condition is used in the Stone-Weierstrass thm.