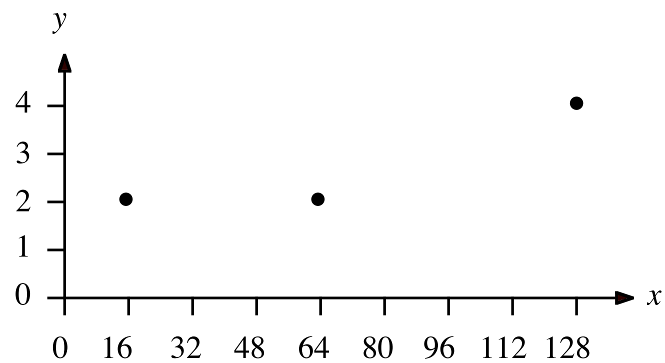
Ex: Find a way to fit three data points, with an exponential curve:

$$y = f(\vec{a},x) = a_1 e^{a_2 x}$$

where

$(x_1, y_1) = (16, 2)$ $(x_2, y_2) = (64, 2)$ $(x_1, y_1) = (128, 4)$



Here, we have $f(\vec{a},\vec{x}_i) = a_1 e^{a_2 x_i}$ since $\vec{x}_i = x_i$ is one-dimensional.

We proceed by calculating derivatives:

$$f_{1i} \equiv \left.\frac{df(\vec{a},\vec{x}_i)}{da_1}\right|_{\vec{x}=x_i} = \frac{d}{da_1} a_1 e^{a_2 \vec{x}}\Big|_{\vec{x}=x_i} = e^{a_2 x_i}$$

$$f_{2i} \equiv \left.\frac{df(\vec{a},\vec{x})}{da_2}\right|_{\vec{x}=x_i} = \frac{d}{da_2} a_1 e^{a_2 \vec{x}}\Big|_{\vec{x}=x_i} = a_1 x_i e^{a_2 x_i}$$

Next, we solve $\sum_{i=1}^{N} y_i f_{ji} = \sum_{i=1}^{N} f_i f_{ji}, \quad j = 1,2:$

$$\sum_{i=1}^{N=3} y_i e^{a_2 x_i} = \sum_{i=1}^{N=3} a_1 e^{a_2 x_i} e^{a_2 x_i}$$

and

$$\sum_{i=1}^{N=3} y_i a_1 x_i e^{a_2 x_i} = \sum_{i=1}^{N=3} a_1 e^{a_2 x_i} a_1 x_i e^{a_2 x_i}$$

or

$$\sum_{i=1}^{N=3} y_i e^{a_2 x_i} = \sum_{i=1}^{N=3} a_1 e^{2a_2 x_i}$$

and

$$\sum_{i=1}^{N=3} y_i a_1 x_i e^{a_2 x_i} = \sum_{i=1}^{N=3} a_1^2 x_i e^{2a_2 x_i}$$

At this point, the idea is to take the coefficients, a_1 and a_2, outside the summations. Here, we can take a_1 outside:

$$\sum_{i=1}^{N=3} y_i e^{a_2 x_i} = a_1 \sum_{i=1}^{N=3} e^{2a_2 x_i}$$

and

$$a_1 \sum_{i=1}^{N=3} y_i x_i e^{a_2 x_i} = a_1^2 \sum_{i=1}^{N=3} x_i e^{2a_2 x_i}$$

Our goal is to have summations involving only the data. Unfortunately, we are unable to extract a_2 from the summations. Thus, this direct approach fails to yield a simple solution. We might try to use an iterative method, but this adds considerable complexity.

An alternative approach is to take logarithms of data points and the model function:

$$\ln y_i = f(\vec{a}, x_i) = \ln a_1 e^{a_2 x_i} = \ln a_1 + a_2 x_i$$

NOTE: We have chosen to change the problem we are solving so we can apply the least-squares method in a straightforward way.

We set derivatives to zero:

$$\frac{\partial E}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial E}{\partial a_2} = 0$$

We proceed by calculating the derivatives:

$$0 = \frac{\partial E}{\partial a_1} = \frac{d}{da_1}\sum_{i=1}^{3}[\ln y_i - (\ln a_1 + a_2 x_i)]^2 = \sum_{i=1}^{3}\frac{d}{da_1}[\ln y_i - (\ln a_1 + a_2 x_i)]^2$$

$$= -\sum_{i=1}^{3}[\ln y_i - (\ln a_1 + a_2 x_i)]\frac{1}{a_1}$$

and

$$0 = \frac{\partial E}{\partial a_2} = \frac{d}{da_2}\sum_{i=1}^{3}[\ln y_i - (\ln a_1 + a_2 x_i)]^2 = \sum_{i=1}^{3}\frac{d}{da_2}[\ln y_i - (\ln a_1 + a_2 x_i)]^2$$

$$= -\sum_{i=1}^{3}[\ln y_i - (\ln a_1 + a_2 x_i)]a_2$$

We can now take coefficients outside of the summations:

$$\frac{1}{a_1}\sum_{i=1}^{N=3}\ln y_i = \frac{\ln a_1}{a_1}\sum_{i=1}^{N=3}1 + \frac{a_2}{a_1}\sum_{i=1}^{N=3}x_i \quad \text{or} \quad \sum_{i=1}^{N=3}\ln y_i = (\ln a_1)\sum_{i=1}^{N=3}1 + a_2\sum_{i=1}^{N=3}x_i$$

and

$$\sum_{i=1}^{N=3}(\ln y_i)x_i = (\ln a_1)\sum_{i=1}^{N=3}x_i + a_2\sum_{i=1}^{N=3}x_i^2$$

We define α_1 to simplify the appearance of our equations:

$$\alpha_1 \equiv \ln a_1$$

Writing the equations in matrix form, we have the following result:

$$\begin{bmatrix} 3 = \sum_{i=1}^{N=3}1 & \sum_{i=1}^{N=3}x_i \\ \sum_{i=1}^{N=3}x_i & \sum_{i=1}^{N=3}x_i^2 \end{bmatrix}\begin{bmatrix}\alpha_1 \\ \mathrm{a}_2\end{bmatrix} = \begin{bmatrix}\sum_{i=1}^{N=3}\ln y_i \\ \sum_{i=1}^{N=3}(\ln y_i)x_i\end{bmatrix}$$

or in symbolic form:

$$X \qquad \vec{a} \quad = \quad \vec{d}$$

Our solution in matrix notation is obtained by multiplying both sides by X^{-1}:

$$\vec{a} = X^{-1}\vec{d}$$

Using the numerical data provided in the problem, we have the following calculation:

$$\begin{bmatrix} 3 & 208 \\ 208 & 20736 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4\ln 2 \\ 336\ln 2 \end{bmatrix}$$

or

$$\begin{bmatrix} \alpha_1 \\ a_2 \end{bmatrix} = \frac{1}{3(20736) - 208^2} \begin{bmatrix} 20736 & -208 \\ -208 & 3 \end{bmatrix} \begin{bmatrix} 4\ln 2 \\ 336\ln 2 \end{bmatrix}$$

or

$$\begin{bmatrix} \alpha_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{3(20736) - 208^2}(20736 * 4 - 208 * 336)\ln 2 \\ \dfrac{1}{3(20736) - 208^2}(-208 * 4 + 3 * 336)\ln 2 \end{bmatrix}$$

or

$$\begin{bmatrix} \alpha_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.689\ln 2 \\ 0.00929\ln 2 \end{bmatrix} = \begin{bmatrix} 0.478 \\ 0.00644 \end{bmatrix}$$

or

$$a_1 = e^{\alpha_1} = 1.61, \; a_2 = 0.00644$$

or

$$y = 1.61 e^{0.00644 x}$$

For the data points, we get the following y values:

$(x_1 = 16, \hat{y}_1) = 1.78$ $\quad$ $(x_2 = 64, \hat{y}_2) = 2.43$ $\quad$ $(x_3 = 128, \hat{y}_3) = 3.67$