EX: Find a multiple regression (i.e., hyperplane fit) to the following data:

(3, 4, 27) (5, 5, 15) (2, 6, 23) (1, 3, 9) (4, 2, 1)

SOL'N: The hyperplane is defined as follows:

$$y = (b, a_1, a_2) \circ (1, x_1, x_2) = b + a_1 x_1 + a_2 x_2$$

We write matrix terms for the data and hyperplane fit:

$$X = \begin{bmatrix} 1 & x_{11} & x_{21} \\ 1 & x_{12} & x_{22} \\ 1 & x_{13} & x_{23} \\ 1 & x_{14} & x_{24} \\ 1 & x_{15} & x_{25} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 5 & 5 \\ 1 & 2 & 6 \\ 1 & 1 & 3 \\ 1 & 4 & 2 \end{bmatrix} \qquad (b, \vec{a}) = \begin{bmatrix} b \\ a_1 \\ a_2 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 27 \\ 15 \\ 23 \\ 9 \\ 1 \end{bmatrix}$$

The matrix equation for the data and hyperplane fit are as follows:

$$X \bullet (b, \vec{a}) = \vec{y}$$

A pseudoinverse of X yields the least-squares solution:

$$(b, \vec{a}) = X^+ \vec{y}$$

where

$$X^+ = (X^T X)^{-1} X^T$$

NOT'N: $X^T \equiv X$ transpose

$$X^+ = \begin{bmatrix} 0.2 & -0.8 & -0.3 & 1.2 & 0.7 \\ 0.0 & 0.2 & -0.1 & -0.2 & 0.1 \\ 0.0 & 0.1 & 0.2 & -0.1 & -0.2 \end{bmatrix}$$

$$(b, \vec{a}) = X^+ \vec{y} = (-2, -1, 5)$$

Plugging the (x_1, x_2) coordinates into the hyperplane equation yields the following estimated y values:

$$\hat{\vec{y}} = X \bullet (b, \vec{a})$$

or

$$[\vec{x}_1, \vec{x}_2, \hat{\vec{y}}] = \begin{bmatrix} 3 & 4 & 15 \\ 5 & 5 & 18 \\ 2 & 6 & 26 \\ 1 & 3 & 12 \\ 4 & 2 & 4 \end{bmatrix}$$

The errors are found by subtracting y values from estimated y values:

$$\vec{e} = \begin{bmatrix} 15 \\ 18 \\ 26 \\ 12 \\ 4 \end{bmatrix} - \begin{bmatrix} 27 \\ 15 \\ 23 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

NOTE: The errors sum to zero, but this is not always the case.

In the plot, below, the estimates are shown as circles: