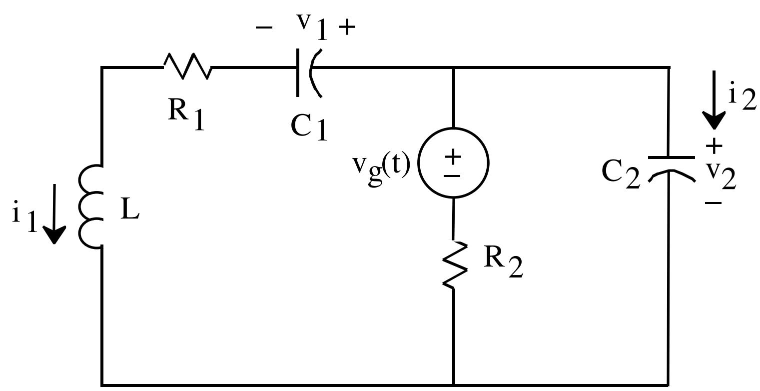
By: Carl H. Durney and Neil E. Cotter

EX: Write the state-variable equations for the circuit shown below.



The voltage $v_g(t)$ changes instantly from $-v_o$ to v_o at $t = 0$.

ANS:

$$di_1/dt = (v_2 - v_1)/L - i_1 R_1/L$$

$$dv_1/dt = i_1/C_1$$

$$dv_2/dt = -i_1/C_2 + (v_o - v_2)/(R_2 C_2)$$

SOL'N: The state variables are always the inductor currents and capacitor voltages (which are also the variables we use to calculate stored energy). Thus, our state variables are i_{L1}, v_{C1}, and v_{C2}. We denote these as i_1, v_1, and v_2.

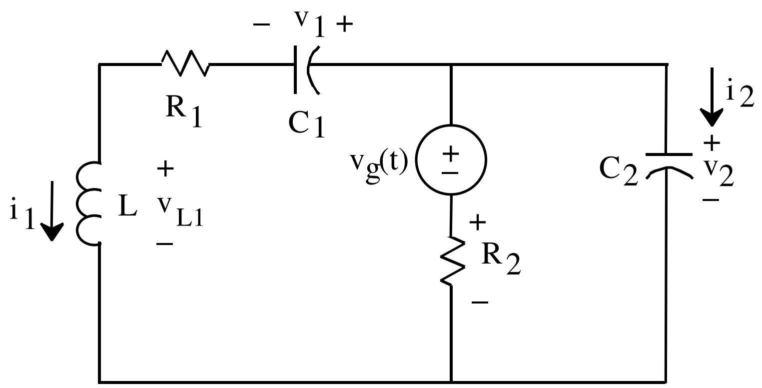
We use the basic component equations to translate derivatives of state variables into non-derivatives:

$$\frac{di_L}{dt} = \frac{v_L}{L}$$

$$\frac{dv_C}{dt} = \frac{i_C}{C}$$

Application of these equations reduces the problem to that of writing equations for v_{L1}, i_{C1}, and i_{C2}. Each of these equations must have only the state variables, i_{L1}, v_{C1}, and v_{C2}, on the other side so the final equations (in terms of the derivatives of state variables) involve only state variables.

The circuit diagram below shows v_{L1}, v_{L2}, and i_{C1}. We now apply Kirchhoff's laws—voltage loops and current sums at nodes—to find our state-space equations.



The equation for v_{L1} must come from a voltage loop, and the voltage loop around the outside will suffice in this case. We use Ohm's law to express the voltages across R_1.

$$v_{L1} + i_1 R_1 + v_1 - v_2 = 0V \text{ or } v_{L1} = v_2 - v_1 - i_1 R_1$$

(Note that the inner voltage loop that includes L_1 would pose difficulties with expressing the voltage for R_2. We could express the voltage across R_2 as $v_2 - v_g(t)$, however, and obtain the same equation as above.)

The equation for i_{C1} is simple:

$$i_{C1} = i_1$$

The equation for i_{C2} must come from a current summation. In this circuit, there are only two nodes: the top and the bottom rails. Because they effectively yield the same current summation equation, we only use one of these nodes. For the top node, summing currents flowing out of the node poses the problem of writing the current in the middle branch in terms of state variables. The solution is to the inner voltage loop on the right side to solve for the current through R_2:

$$i_{R2} = \frac{v_2 - v_g(t)}{R_2}$$

Using this equation for i_{R2} and summing currents yields an equation for i_{C2}:

$$i_1 + \frac{v_2 - v_g(t)}{R_2} + i_{C2} = 0A \text{ or } i_{C2} = -i_1 + \frac{-v_2 + v_g(t)}{R_2}$$

To complete the derivation, we use the basic component equations to change v_{L1}, v_{L2}, and i_{C1} back into derivatives of state variables.

$$\frac{di_1}{dt} = \frac{v_2 - v_1 - i_1 R_1}{L_1}$$

$$\frac{dv_1}{dt} = \frac{i_1}{C_1}$$

$$\frac{dv_2}{dt} = \frac{-i_1 R_2 + v_o - v_2}{R_2 C}$$

Note that in the third equation we have substituted the value of v_o for v_g for $t > 0$.