THM: Given n Bernoulli trials with probability of success for each trial being p, the probability, $P(m \text{ of } n)$, of exactly m successes in n trials approaches the probability density of $x = m$ for a normal (i.e., gaussian) distribution with $\mu = np$ and $\sigma^2 = npq$:

$$\text{As } n \to \infty,\ P(m \text{ of } n) \to f(x = m) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(m-\mu)^2/2\sigma^2}.$$

PROOF: We follow the general method of proof given in [1].

For Bernoulli trials we have the following value for $P(m \text{ of } n)$:

$$P(m \text{ of } n) = {}_nC_m \cdot p^m q^{n-m}$$

where ${}_nC_m \equiv \dfrac{n!}{(n-m)!m!}$ is the combinatoric coefficient.

For the proof, we consider different values of n, and we will consider m to be a fixed number, k, of standard deviations from the mean as n increases.

$$m = \mu + k\sigma$$

NOTE: Although m is an integer, the method of proof allows k to have any real value.

We use Stirling's formula, [2], to approximate the factorials in ${}_nC_m$:

$$n! = \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n+\frac{\theta}{12n}}$$

where $n > 0$ and $0 < \theta < 1$.

NOTE: Stirling's formula is related to the Stirling series expansion of the gamma function in powers of $1/n$, (see [3]). The Stirling series has the curious property that it produces very accurate approximations of the gamma functions with only a few terms—and actually diverges if all the terms are used.

Using Stirling's formula for the terms of ${}_nC_m$, yields the following expression:

$${}_nC_m = \frac{\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n+\frac{\theta_1}{12n}}}{\sqrt{2\pi}\,(n-m)^{n-m+\frac{1}{2}} e^{-n-m+\frac{\theta_2}{12(n-m)}} \sqrt{2\pi}\, m^{m+\frac{1}{2}} e^{-m+\frac{\theta_3}{12m}}}$$

As n becomes large, so do $n - m$ and m, and the residual terms involving θ_1, θ_2, and θ_3 become vanishingly small. Thus, we may eliminate the θ terms and, after also canceling common factors of $\sqrt{2\pi}$ and the exponentials of e, write the following expression:

$$_nC_m \to \frac{n^{n+\frac{1}{2}}}{\sqrt{2\pi}\,(n-m)^{n-m+\frac{1}{2}}\,m^{m+\frac{1}{2}}} \text{ as } n \to \infty$$

If we split the n^n term into two pieces in the numerator, we can match up the exponents in the numerator and denominator:

$$_nC_m \to \frac{n^{n-m}n^m\sqrt{n}}{\sqrt{2\pi}\,(n-m)^{n-m}\sqrt{\text{n - m}}\cdot m^m\sqrt{m}} \text{ as } n \to \infty$$

or

$$_nC_m \to \frac{\sqrt{n}}{\sqrt{2\pi(n-m)m}}\left(\frac{n}{n-m}\right)^{n-m}\left(\frac{n}{m}\right)^m \text{ as } n \to \infty$$

Now we invert the terms being exponentiated and use the following formulas:

$$\frac{m}{n} = \frac{\mu + k\sigma}{n} = \frac{np + k\sigma}{n} = p\left(1 + \frac{k\sigma}{np}\right)$$

and

$$\frac{n-m}{n} = 1 - \frac{\mu + k\sigma}{n} = 1 - \frac{np + k\sigma}{n} = q\left(1 - \frac{k\sigma}{nq}\right)$$

Substituting these expressions yields the following equation:

$$_nC_m \to \frac{\sqrt{n}}{\sqrt{2\pi(n-m)m}}\left(q\left(1 - \frac{k\sigma}{nq}\right)\right)^{-n+m}\left(p\left(1 + \frac{k\sigma}{np}\right)\right)^{-m} \text{ as } n \to \infty$$

The terms having n in their denominators will become small as n becomes large. Thus, we use an approximation that exploits this behavior:

$$\ln(1 + x) \approx x - \frac{x^2}{2} \text{ for } x \text{ small}$$

or

$$(1+x)^r = e^{r\ln(1+x)} \approx e^{r\left(x - \frac{x^2}{2}\right)} \text{ for } x \text{ small (from Taylor series for ln)}$$

Applying this identity to our formula for the combinatoric coefficient, we have the following expression:

$$_nC_m \to \frac{\sqrt{n}}{\sqrt{2\pi(n-m)m}} q^{-n+m} p^{-m} e^{(-n+m)\left(-k\frac{\sigma}{nq} - \frac{k^2\sigma^2}{2n^2q^2}\right)} e^{-m\left(k\frac{\sigma}{np} - \frac{k^2\sigma^2}{2n^2p^2}\right)}$$

Using $m = np + k\sigma$ and $m - n = -nq + k\sigma$ we have

$$_nC_m \to \frac{\sqrt{n}}{\sqrt{2\pi(n-m)m}} q^{-n+m} p^{-m} e^{(k\sigma - nq)\left(-k\frac{\sigma}{nq} - \frac{k^2\sigma^2}{2n^2q^2}\right) - (k\sigma + np)\left(k\frac{\sigma}{np} - \frac{k^2\sigma^2}{2n^2p^2}\right)}$$

If we consider just the exponent, we have the following calculation:

$$-\frac{k^2\sigma^2}{nq} - \frac{k^2\sigma^2}{np} + k\sigma\left(\frac{k^2\sigma^2}{2n^2p^2} - \frac{k^2\sigma^2}{2n^2q^2}\right) + nq\frac{k^2\sigma^2}{2n^2q^2} + np\frac{k^2\sigma^2}{2n^2p^2}$$

$$= -\frac{k^2\sigma^2 p}{npq} - \frac{k^2\sigma^2 q}{npq} + k\sigma\left(\frac{k^2\sigma^2 q^2}{2n^2p^2q^2} - \frac{k^2\sigma^2 p^2}{2n^2p^2q^2}\right) + (np + nq)\frac{k^2\sigma^2}{2n^2pq}$$

Using $\sigma^2 = npq$ the simplification of the exponent continues:

$$= -k^2 p - k^2 q + k\sigma\left(\frac{k^2q^2}{2\sigma^2} - \frac{k^2p^2}{2\sigma^2}\right) + \frac{k^2}{2}$$

$$= -k^2 + k\left(\frac{k^2q^2}{2\sigma} - \frac{k^2p^2}{2\sigma}\right) + \frac{k^2}{2}$$

$$= -\frac{k^2}{2} + k\left(\frac{k^2q^2}{2\sigma} - \frac{k^2p^2}{2\sigma}\right)$$

We observe that the second term is proportional to $1/\sqrt{n}$ and vanishes as n becomes large. Dropping this term yields the following expression:

$$ {}_nC_m \to \frac{\sqrt{n}}{\sqrt{2\pi(n-m)m}} q^{-n+m} p^{-m} e^{-\frac{k^2}{2}} \text{ as } n \to \infty $$

If we now multiply by the probability, $p^m q^{n-m}$ of one particular pattern of m successes occurring, we obtain the following expression:

$$ P(m \text{ of } n) \to \frac{\sqrt{n}}{\sqrt{2\pi(n-m)m}} e^{-\frac{k^2}{2}} \text{ as } n \to \infty $$

We have the following simplification for the factor in front:

$$ \sqrt{\frac{n}{2\pi(n-m)m}} = \sqrt{\frac{1}{2\pi(1-\frac{m}{n})m}} = \sqrt{\frac{1}{2\pi(1-\frac{np+k\sigma}{n})(np+k\sigma)}} $$

For n large, $k\sigma$ is much smaller than n, leading to the following result:

$$ \sqrt{\frac{n}{2\pi(n-m)m}} \approx \sqrt{\frac{1}{2\pi(1-\frac{np}{n})(np)}} = \sqrt{\frac{1}{2\pi qnp}} = \sqrt{\frac{1}{2\pi\sigma^2}} $$

With this substitution, and using $k^2 = \dfrac{(m-\mu)^2}{\sigma^2}$ we complete our proof:

$$ P(m \text{ of } n) \to \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{k^2}{2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(m-\mu)^2/2\sigma^2} \text{ as } n \to \infty $$

REF: [1] Eugene Lukacs, *Probability and Mathematical Statistics, an Introduction*, New York, NY: Academic Press, 1972.

[2] Milton Abramowitz and Irene A. Stegun, Eds., *Handbook of Mathematical Functions: National Bureau of Standards Applied Mathematics Series 55*, Washington, D.C.: Government Printing Office, 1972.

[3] Carl M. Bender and Steven A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers,* New York, NY: McGraw-Hill, 1978.