TOOL: For samples from a normal (or gaussian) distribution with mean μ and variance σ^2, the distribution of the sample mean, $\overline{X}$, for n samples is normal (or gaussian) with the following mean and variance:

$$\mu_{\overline{X}} = \mu \qquad \text{and} \qquad \sigma_{\overline{X}}^2 = \frac{\sigma^2}{n}$$

It follows that random variable Z defined as follows has a standard normal (or gaussian) distribution:

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

PROOF: The sample mean is the average of n independent samples.

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}X_1 + \ldots + \frac{1}{n}X_n$$

Since the sample mean is a linear combination of independent samples, it follows that the mean value of $\overline{X}$ is the linear combination of the mean values of the X_i.

$$\mu_{\overline{X}} = \frac{1}{n}\mu + \ldots + \frac{1}{n}\mu = \frac{n}{n}\mu = \mu$$

Since the sample mean is a linear combination of independent samples, it also follows that the variance of the sample mean is a sum of variances of the X_i each multiplied by the square of their coefficient.

$$\sigma_{\overline{X}}^2 = \left(\frac{1}{n}\right)^2 \sigma^2 + \ldots + \left(\frac{1}{n}\right)^2 \sigma^2 = \frac{n}{n^2}\sigma^2 = \frac{1}{n}\sigma^2$$

Finally, we note that a linear combination of independent normal (or gaussian) random variables is a normal (or gaussian) random variable.