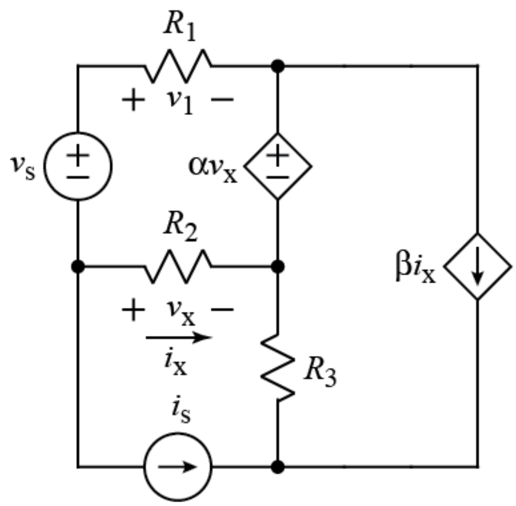
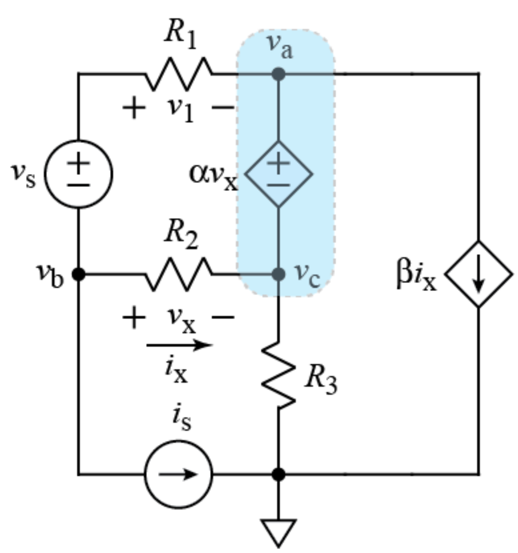
EX: Using superposition, find an expression for v_1 in the circuit shown below.



SOL'N I: We first illustrate brute force attempts at solution using the node-voltage method and superposition. When the suffering is complete, we consider judicious use of the reference placement, superposition, and Kirchhoff's laws to simplify matters.

For the node-voltage method, we label a reference and any nodes where three or more components are joined.



Before going any further, we define variables for dependent sources in terms of node voltages.

$$v_x = v_b - v_c$$

$$i_x = \frac{v_x}{R_2} = \frac{v_b - v_c}{R_2}$$

We use the above definitions whenever we write v_x or i_x.

The blue indicates a supernode. We proceed to sum currents out of nodes, starting with the supernode, (two nodes connected by just a voltage source), for which we sum currents out of the blue bubble. The key is to define currents in terms of node voltages.

$$\frac{v_a - (v_b + v_s)}{R_1} + \beta\frac{v_b - v_c}{R_2} + \frac{v_c - v_b}{R_2} + \frac{v_c}{R_3} = 0\,\text{A}$$

or

$$\frac{v_a - (v_b + v_s)}{R_1} + (\beta - 1)\frac{v_b - v_c}{R_2} + \frac{v_c}{R_3} = 0\,\text{A}$$

or, if we write things in a form suitable for matrix solution,

$$\frac{1}{R_1}v_a + \left(-\frac{1}{R_1} + \frac{\beta-1}{R_2}\right)v_b + \left(\frac{1-\beta}{R_2} + \frac{1}{R_3}\right)v_c = \frac{v_s}{R_1} \tag{1}$$

The voltage equation is simpler:

$$v_a - v_c = \alpha(v_b - v_c).$$

or, in matrix compatible form,

$$(1)v_a + (-\alpha)v_b + (\alpha - 1)v_c = 0\,\text{V}. \tag{2}$$

Finally, we have the v_b node:

$$\frac{v_b + v_s - v_a}{R_1} + \frac{v_b - v_c}{R_2} + i_s = 0\,\text{A}$$

or, in matrix compatible form,

$$\left(-\frac{1}{R_1}\right)v_a + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_b + \left(-\frac{1}{R_2}\right)v_c = -\frac{v_s}{R_1} - i_s. \tag{3}$$

Now what? We solve the equations by hand, introducing terms to save writing:

$$X \equiv \frac{R_1(1-\beta)}{R_2} \quad \text{and} \quad Y \equiv \frac{R_1}{R_2}$$

Multiplying (1) by R_1 and (3) by $-R_1$ gives us the following three equations:

$$(1)v_a + (-1-X)v_b + (X + R_1/R_3)v_c = v_s \tag{1'}$$

$$(1)v_a + (-\alpha)v_b + (\alpha - 1)v_c = 0\,\text{V} \tag{2'}$$

$$(1)\text{v}_a + (-1-Y)v_b + (Y)v_c = v_s + i_s R_1 \tag{3'}$$

Solving (2') for v_a, we may substitute the result into (1') and (3').

$$(1)v_a = (\alpha)v_b + (1-\alpha)v_c \tag{2''}$$

which gives

$$\alpha v_b + (1-\alpha)v_c + (-1-X)v_b + (X + R_1/R_3)v_c = v_s \tag{1''}$$

and

$$\alpha v_b + (1-\alpha)v_c + (-1-Y)v_b + (Y)v_c = v_s + i_s R_1 \tag{3''}$$

Cleaning things up, we have

$$(\alpha - 1 - X)v_b + (1 - \alpha + X + R_1/R_3)v_c = v_s \tag{1'''}$$

and

$$(\alpha - 1 - Y)v_b + (1 - \alpha + Y)v_c = v_s + i_s R_1 \tag{3'''}$$

We can do a bit more with (1''') and (3'''):

$$(\alpha - 1 - X)v_b + (-\alpha + 1 + X + R_1/R_3)v_c = v_s$$

or

$$v_b + \left(-1 + \frac{R_1/R_3}{\alpha - 1 - X}\right)v_c = \frac{v_s}{\alpha - 1 - X} \tag{1''''}$$

and

$$(\alpha - 1 - Y)v_b + (-\alpha + 1 + Y)v_c = v_s + i_s R_1$$

or

$$v_b + (-1)v_c = \frac{v_s + i_s R_1}{\alpha - 1 - Y} \qquad (3'''')$$

Solving (3'''') for v_c, we may substitute into (1''''), leaving only v_b. Keeping v_b instead of v_c is convenient since we want to find v_1.

$$v_c = v_b - \frac{v_s + i_s R_1}{\alpha - 1 - Y}$$

Substituting into (1''''), we have

$$v_b + \left(-1 + \frac{R_1 / R_3}{\alpha - 1 - X}\right)\left(v_b - \frac{v_s + i_s R_1}{\alpha - 1 - Y}\right) = \frac{v_s}{\alpha - 1 - X}$$

or

$$\left(\frac{R_1 / R_3}{\alpha - 1 - X}\right)v_b = \frac{v_s}{\alpha - 1 - X} + \left(-1 + \frac{R_1 / R_3}{\alpha - 1 - X}\right)\left(\frac{v_s + i_s R_1}{\alpha - 1 - Y}\right)$$

Ugh! We struggle on... Multiply both sides by $\alpha - 1 - X$.

$$(R_1 / R_3)v_b = v_s + \big(-(\alpha - 1 - X) + R_1 / R_3\big)\left(\frac{v_s + i_s R_1}{\alpha - 1 - Y}\right)$$

or

$$(R_1 / R_3)v_b = \left(\frac{\alpha - 1 - Y - (\alpha - 1 - X) + R_1 / R_3}{\alpha - 1 - Y}\right)v_s + \big(-(\alpha - 1 - X) + R_1 / R_3\big)\left(\frac{i_s R_1}{\alpha - 1 - Y}\right)$$

or

$$(R_1 / R_3)v_b = \left(\frac{X - Y + R_1 / R_3}{\alpha - 1 - Y}\right)v_s + \big(X + 1 - \alpha + R_1 / R_3\big)\left(\frac{i_s R_1}{\alpha - 1 - Y}\right)$$

Since $X - Y = \beta R_1/R_2$, if we multiply by R_3/R_1, we have

$$v_{\rm b} = \left(\frac{\beta R_3 / R_2 + 1}{\alpha - 1 - Y} \right) v_{\rm s} + \big((X + 1 - \alpha) R_3 + R_1 \big) \left(\frac{i_{\rm s}}{\alpha - 1 - Y} \right). \qquad (1''''')$$

Very well, but we want v_1. Curiously, things are much better when we try to find v_1.

$$v_1 = v_b + v_s - v_a$$

where

$$(1)v_{\rm a} = (\alpha)v_{\rm b} + (1 - \alpha)v_{\rm c} \qquad (2'')$$

so

$$v_1 = v_b + v_s - \alpha v_{\rm b} - (1 - \alpha)v_{\rm c} = v_s + (1 - \alpha)(v_{\rm b} - v_{\rm c})$$

where

$$v_{\rm b} + (-1)v_{\rm c} = \frac{v_s + i_{\rm s} R_1}{\alpha - 1 - Y} \qquad (3'''')$$

so

$$v_1 = v_s + (1 - \alpha)\frac{v_{\rm s} + i_s R_1}{\alpha - 1 - Y} = v_s + (1 - \alpha)\frac{v_{\rm s} + i_s R_1}{\alpha - 1 - R_1 / R_2}$$

or

$$v_1 = \frac{\alpha - 1 - R_1 / R_2}{\alpha - 1 - R_1 / R_2} v_s + (1 - \alpha)\frac{v_{\rm s} + i_s R_1}{\alpha - 1 - R_1 / R_2}$$

or

$$v_1 = \frac{\alpha - 1 - R_1 / R_2 + 1 - \alpha}{\alpha - 1 - R_1 / R_2} v_s + (1 - \alpha)\frac{i_s R_1}{\alpha - 1 - R_1 / R_2}$$

or

$$v_1 = \frac{-R_1 / R_2}{\alpha - 1 - R_1 / R_2} v_s + (1 - \alpha)\frac{i_s R_1}{\alpha - 1 - R_1 / R_2}$$

or

$$v_1 = \frac{-R_1}{R_2(\alpha - 1) - R_1} v_s + (1-\alpha)\frac{i_s R_1 R_2}{R_2(\alpha - 1) - R_1}$$

or

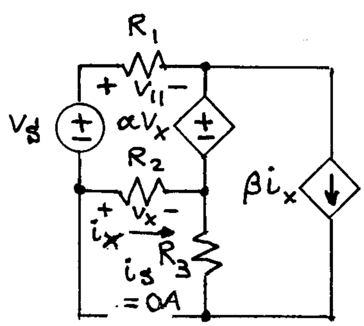
$$v_1 = \frac{v_s R_1 - i_s R_1 R_2 (1-\alpha)}{R_2(1-\alpha) + R_1}$$

Wait. What happened to the mess we had before in (1"")? The above solution didn't even use β! Maybe there is a simpler way to have this problem.

SOL'N II: We use superposition and some basic ideas, like Kirchhoff's laws.

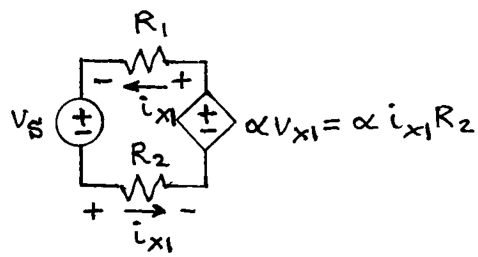
We turn on one independent source at a time. Dependent sources stay on.

case I: v_s on, i_s off = open



If we examine where i_x goes, we discover that i_x flows thru R_1. Thus, a v-loop around the upper left yields a value for i_x.

Note that $v_{x1} = i_{x1} R_2$.



v-loop: $v_S + i_{x1}R_1 - i_{x1}\alpha R_2 + i_{x1}R_2 = 0\text{ V}$

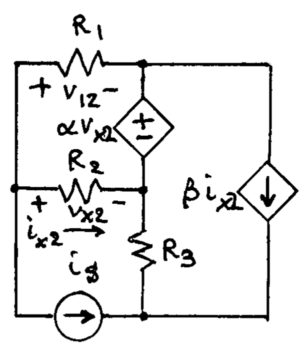
or $i_{x1}(R_1 + R_2 - \alpha R_2) = -v_S$

or $i_{x1} = \dfrac{-v_S}{R_1 + R_2 - \alpha R_2}$

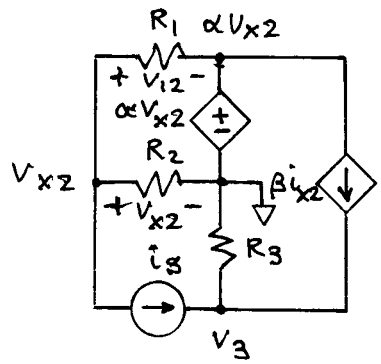
Using Ohm's law to find v_{11} we have

$$v_{11} = -i_{x1}R_1 = v_S \frac{R_1}{R_1 + R_2 - \alpha R_2}$$

case II: v_S off = wire, i_S on



We can always use node-voltage. Putting a reference in the center is convenient.



Consider the v_{x2} node:

$$\frac{v_{x2} - \alpha v_{x2}}{R_1} + \frac{v_{x2}}{R_2} + i_s = 0\text{A}$$

or $$v_{x2}\left(\frac{1}{R_1} - \frac{\alpha}{R_1} + \frac{1}{R_2}\right) = -i_s$$

or $$v_{x2}\left(R_2 - \alpha R_2 + R_1\right) = -i_s R_1 R_2$$

or $$v_{x2} = \frac{-i_s R_1 R_2}{R_1 + R_2(1-\alpha)}$$

$$v_{12} = v_{x2} - \alpha v_{x2} = (1-\alpha)\, v_{x2}$$

or $$v_{12} = \frac{-i_s R_1 R_2 (1-\alpha)}{R_1 + R_2(1-\alpha)}$$

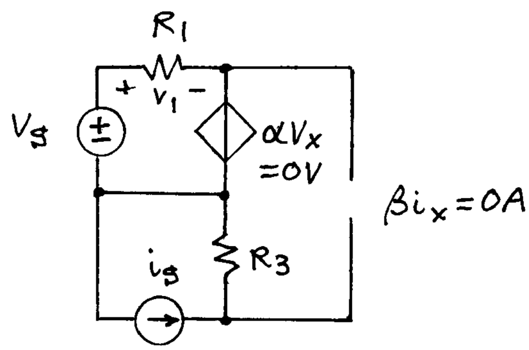
Now we sum the results from the two cases.

$$\boxed{v_1 = v_{11} + v_{12} = \frac{v_s R_1 - i_s R_1 R_2 (1-\alpha)}{R_1 + R_2(1-\alpha)}}$$

Consistency checks follow.

We perform some consistency checks. We set some component values to zero to create a circuit with an obvious solution. Then we see if our above expression for v_1 gives the correct answer.

Check 1: Set $R_2 = 0$ so $v_x = 0\,V$.
Set $\beta = 0$ so $\beta i_x = 0A$.



From the v-loop in the upper left we have $v_1 = v_S$.

Our answer above gives

$$v_1 = \frac{v_S \cancel{R_1} - i_S \cancel{R_1 (0)} (1-\alpha)}{\cancel{R_1} + \cancel{0(1-\alpha)}} = v_S \checkmark$$

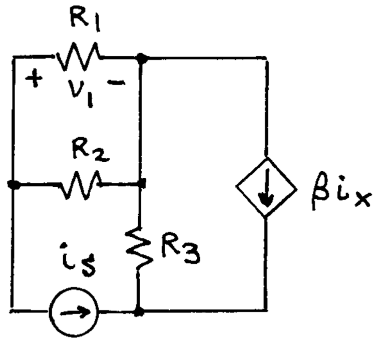
Check 2: Set $R_1 = 0$, then $v_1 = 0\,V$.

Our answer for v_1 gives

$$v_1 = \frac{v_S(0) - i_S(0) R_2 (1-\alpha)}{0 + R_2(1-\alpha)} = 0\,V \checkmark$$

One more check follows.

Check 3: Set $v_s = 0\,\text{V}$ and $\alpha = 0$.



Careful inspection reveals that R_1 and R_2 are in parallel and i_s flows thru R_1 and R_2.

So we have a current divider, and voltage v_1 is given by

$$v_1 = -i_s \frac{R_2}{R_1 + R_2} \cdot R_1$$

Our answer above gives

$$v_1 = \frac{\cancelto{0}{0 \cdot R_1} - i_s R_1 R_2 (1-0)}{R_1 + R_2(1-0)}$$

or $$v_1 = \frac{-i_s R_1 R_2}{R_1 + R_2} \quad \checkmark$$

All checks thus far are satisfied.