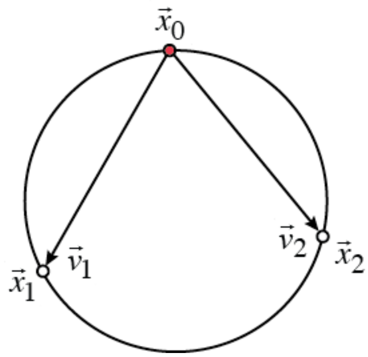
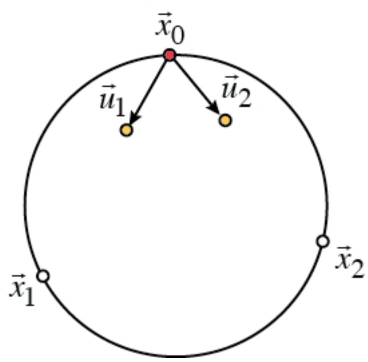
TOOL: The following algorithm finds the center point of an N-dimensional sphere given $N + 1$ points, $\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_N$, and is based on the idea that the center of a sphere lies on bisectors of line segments connecting points on the perimeter:

i) Determine vectors, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N$, pointing from one point, $\vec{x}_0$, chosen as an anchor point, toward each other point.



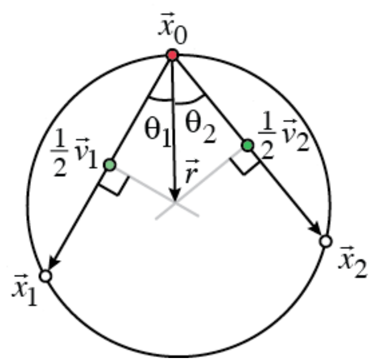
$$\vec{v}_i = \vec{x}_i - \vec{x}_0$$

ii) By dividing by their lengths, normalize the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N$ to create unit-length vectors, $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N$, pointing from $\vec{x}_0$ toward each other point.



$$\vec{u}_i = \frac{\vec{v}_i}{|\vec{v}_i|}$$

iii) Find the vector, $\vec{r}$, whose projection on each unit-length vector, $\vec{u}_i$, has its endpoint at the midpoint of the line segment from $\vec{x}_0$ to $\vec{x}_i$, (i.e. the projection of $\vec{r}$ on $\vec{v}_i$ equals $\vec{v}_i / 2$). The projection of $\vec{r}$ on $\vec{v}_i$ is given by the dot product of $\vec{r}$ and $\vec{u}_i$.



$$\vec{r} \circ \vec{u}_i = |\vec{r}||\vec{u}_i|\cos\theta_i = |\vec{r}|\cos\theta_i = \tfrac{1}{2}|\vec{v}_i|$$

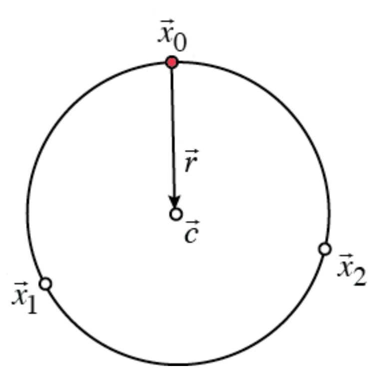
Group these equations to yield a matrix formula for $\vec{r}$.

$$\begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vdots \\ \vec{u}_N^T \end{bmatrix} \vec{r} = \begin{bmatrix} \frac{1}{2}|\vec{v}_1| \\ \frac{1}{2}|\vec{v}_2| \\ \vdots \\ \frac{1}{2}|\vec{v}_N| \end{bmatrix}$$

Since the $\vec{v}_i$ vectors arise from points on a sphere, they are not dependent. Thus, the matrix equation is nonsingular and always solvable.

$$\vec{r} = \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vdots \\ \vec{u}_N^T \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2}|\vec{v}_1| \\ \frac{1}{2}|\vec{v}_2| \\ \vdots \\ \frac{1}{2}|\vec{v}_N| \end{bmatrix}$$

iv) The center point, $\vec{c}$, of the circle is found by summing $\vec{x}_0$ and $\vec{r}$.



$$\vec{c} = \vec{x}_0 + \vec{r}$$