EE 5961-2 / 6961-2 Lecture # 27

20 Mar 2000
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Methods of Creating Wavelets:

Creating scaling funcs by Fourier techniques:

$\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is orthonormal (not complete since is only shifts)

iff $\langle \varphi(t), \varphi(t-n)\rangle = \delta[n] \overset{\mathcal{F}ourier}{\longleftrightarrow} \sum_{k\in\mathbb{Z}} |\Phi(\omega+2k\pi)|^2 = 1$ for all ω

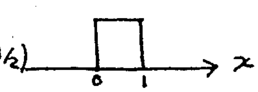
Meyer's Wavelet:
$$\Phi(\omega) = \begin{cases} \theta\left(2+\frac{3\omega}{2\pi}\right) & \omega \le 0 \\ \theta\left(2-\frac{3\omega}{2\pi}\right) & \omega \ge 0 \end{cases}$$

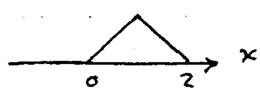
$$\theta(x) = \begin{cases} 0 & x \le 0 \\ 3x^2 - 2x^3 & 0 \le x \le 1 \\ 1 & 1 \le x \end{cases}$$

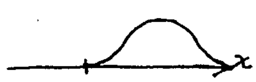
Use standard procedure to create $\psi(t)$ from $\varphi(t)$.

B-splines

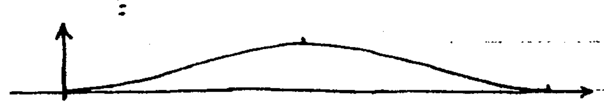
Convolve $\Pi(x)$ with itself N times to get B-spline$_N$:

$\beta_0(t+1/2)$  $\Pi(x)$

$\beta_1(t)$  $\Pi(x) \otimes \Pi(x)$

$\beta_2(t+1/2)$  $\Pi(x) \otimes \Pi(x) \otimes \Pi(x)$

$\vdots$

$\beta_N(t)$  gaussian in limit

for N odd $\quad B_N(\omega) = \left[\frac{\sin(\omega/2)}{\omega/2}\right]^{N+1} \qquad \beta_N(t) = \underbrace{\Pi(x)\otimes\ldots\otimes\Pi(x)}_{N\text{ times}}$

N even $\quad B_N(\omega) = e^{-j\omega/2}\left[\frac{\sin(\omega/2)}{\omega/2}\right]^{N+1} \qquad \beta_N(t+1/2) = \underbrace{\Pi(x)\otimes\ldots\otimes\Pi(x)}_{N\text{ times}}$

Note: shift by 1/2 so nodes of spline at integers.

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B-spline Wavelets (cont)

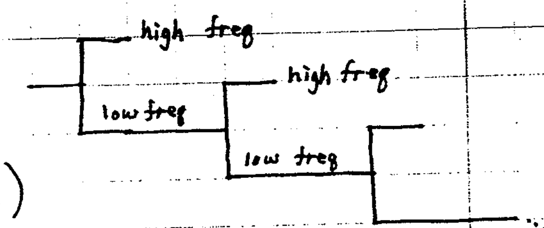
Make an orthonormal set of scaling functions.

Result: $$\Phi(\omega) = \frac{\beta_N(\omega)}{\sqrt{B_{2N+1}(\omega)}}$$

Battle-Lemarié wavelets also based on B-splines with alternative orthonormalization:

$$\Phi(\omega) = \frac{\sin^2(\omega/2)}{(\omega/2)^2\left[1-(2/3)\sin^2(\omega/2)\right]^{1/2}}$$

Iterated Filter Banks



$$\Phi(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right)$$

$$\Psi(\omega) = M_1\left(\frac{\omega}{2}\right) \prod_{k=2}^{\infty} M_0\left(\frac{\omega}{2^k}\right)$$

Need $M_0(0) = 1$ or else $\Pi = 0$ or $\Pi = \infty$

How to find $M_0(\omega)$? Want smoothness ~~&~~ regularity

sufficient condition for convergence: $|M_0(\omega)| > 0$, $|\omega| < \frac{\pi}{2}$.

sufficient condition for continuous $\varphi(t)$:
factor $M_0(\omega) = \left(\frac{1+e^{j\omega}}{2}\right)^N R(\omega)$

$$\text{def } B \equiv \sup_{\omega \in [0, 2\pi]} |R(\omega)|$$

If $B < 2^{N-1}$ then $\varphi(t)$ continuous

Cohen's fixed-pt method

$$\prod_{k=1}^{i} \left|M_0\left(\frac{\omega}{2^k}\right)\right| \Bigg|_{\omega = 2^i\pi/3} = \left|M_0\left(\frac{2\pi}{3}\right)\right|^i$$

So $|M_0(2\pi/3)| \leq 1/2$ or decay not of order $1/\omega$